Day 35 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Double angle equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $2\cos 2\theta + 6\sin\theta - 4 = 0$
for $0 \leq \theta < 2\pi$, giving answers in terms of π.

Use $\cos 2\theta \equiv 1 - 2\sin^2\theta$: $4\sin^2\theta - 6\sin\theta + 2 = 0$
$2(2\sin\theta - 1)(\sin\theta - 1) = 0$
$\sin\theta = \frac{1}{2}$ or $\sin\theta = 1$
Principal values and symmetry in $[0, 2\pi)$
Answer: $\boldsymbol{\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}}$

2 Equations in disguise (substitution) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $x - 3\sqrt{x} - 10 = 0$.

Let $u = \sqrt{x}$: $u^2 - 3u - 10 = 0$.
$(u - 5)(u + 2) = 0$, so $\sqrt{x} = 5$ or $\sqrt{x} = -2$.
$\sqrt{x}$ cannot be negative, so reject $\sqrt{x} = -2$.
$x = 25$.
Answer: $\boldsymbol{x = 25}$

3 Related rates of change (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ladder of length 13 m leans against a vertical wall. Its foot slides away from the wall at 1 m s^{-1}. When the foot is 5 m from the wall, find the rate at which the top is moving down.

$x^2 + y^2 = 169$, so $2x\,dx/dt + 2y\,dy/dt = 0$
When $x = 5$, $y = 12$
$dy/dt = -\frac{x}{y}\,dx/dt = -\frac{5}{12} \times 1$
Answer: $\boldsymbol{\frac{5}{12}\ \text{m s}^{-1}}$

4 Variable force in vector form (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P of mass 5 kg moves under a single force F N, where $\mathbf{F} = (30t + 10)\mathbf{i} + (15t^2 + 10)\mathbf{j}$ and t is the time in seconds. At t = 0, P is at the origin with velocity $(3\mathbf{i} - 2\mathbf{j})$ m s^{-1}.
Find (a) the velocity of P at t = 2, (b) the position vector of P at t = 2.

$\mathbf{a} = \frac{\mathbf{F}}{5} = (6t + 2)\mathbf{i} + (3t^2 + 2)\mathbf{j}$

$\mathbf{v} = \int \mathbf{a}\,dt = (3t^2 + 2t + 3)\mathbf{i} + (t^3 + 2t - 2)\mathbf{j}$ (using $\mathbf{v}(0) = (3\mathbf{i} - 2\mathbf{j})$)
At t = 2: $\mathbf{v} = (19\mathbf{i} + 10\mathbf{j})$
$\mathbf{r} = \int \mathbf{v}\,dt = (t^3 + t^2 + 3t)\mathbf{i} + (0.25t^4 + t^2 - 2t)\mathbf{j}$
At t = 2: $\mathbf{r} = (18\mathbf{i} + 4\mathbf{j})$
Answer: (a) $(19\mathbf{i} + 10\mathbf{j})$ m s^{-1} (b) $(18\mathbf{i} + 4\mathbf{j})$ m

5 Conditional probability: independent binomial counts (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A biased coin lands on 'win' with the same unknown probability p each time. It is used 8 times in round 1 and 8 times in round 2.
Given that exactly 9 wins occur altogether, find the probability that exactly 3 of them were in round 1.

Let $X \sim B(8, p)$, $Y \sim B(8, p)$ be independent; $X + Y \sim B(16, p)$

$P(X = 3 \mid X + Y = 9) = \frac{P(X = 3)P(Y = 6)}{P(X + Y = 9)}$

$= \frac{C(8,3)C(8,6)\,p^9(1-p)^7}{C(16,9)\,p^9(1-p)^7}$

The powers of p and $(1 - p)$ cancel, leaving $\frac{56 \times 28}{11440}$

Answer: $\boldsymbol{\frac{98}{715} = 0.137}$

6 Block on a plank: limiting slip (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A plank of mass 4 kg lies on a smooth horizontal table, with a block of mass 4 kg on it ($\mu = 0.3$ between them). A horizontal force P N acts on the block.
(a) Find the greatest P for which the block does not slip. (b) Find the accelerations of block and plank when P = 47.04.
Take g = 9.8 m s^{-2}.

No slipping: common acceleration $a = \frac{P}{8}$. The plank is accelerated only by friction: $F = 4a$

Limiting friction: $F = \mu mg = 0.3 \times 4 \times 9.8 = 11.76$, so $a = \frac{11.76}{4} = 2.94$

P_max = 8 × 2.94 = 23.52

P = 47.04 > 23.52, so the block slips. Plank: $a = \frac{11.76}{4} = 2.94$

Block: $47.04 - 11.76 = 4a$, so $a = 8.82$

Answer: (a) 23.52 N (b) block 8.82 m s^{-2}, plank 2.94 m s^{-2}

7 Parametric curve and line intersection (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has parametric equations
$x = t$, $y = t^2$. The line $y = 4x - 3$
meets C at A and B. Find the exact length AB.

Substitute: $t^2 = 4t - 3$.
$t^2 - 4t + 3 = 0 \Rightarrow t = 1$ or $t = 3$.
A(1, 1), B(3, 9).
$AB^2 = (-2)^2 + (-8)^2 = 68$.
Answer: AB = $2\sqrt{17}$

8 Stationary points of trigonometric curves (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the coordinates of the stationary points of
$y = 4\cos 2x + 8\cos x$ for $0 < x < 2\pi$,
and determine their nature.

$dy/dx = -8\sin 2x - 8\sin x = -8\sin x(2\cos x + 1)$

dy/dx = 0: $\sin x = 0$ or $\cos x = -\frac{1}{2}$

$x = \pi, \frac{2\pi}{3}, \frac{4\pi}{3}$ in the range

Test the sign of d^2y/dx^2 at each

Answer: $(\frac{2\pi}{3}, -6)$ minimum; $(\pi, -4)$ maximum; $(\frac{4\pi}{3}, -6)$ minimum

9 Normal test: critical region and Type II error (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim N(\mu, 49)$. H_0: $\mu = 47$ is tested against H_1: $\mu < 47$ using the mean of a sample of size 25 at the 10% level.
(a) Find the critical region for the sample mean.
(b) Find the probability of a Type II error when $\mu = 44$.

Under H_0, sample mean $\sim N(47, \frac{49}{25})$, standard error = 1.400

z = 1.2816 for the lower tail: critical value = 47 − 1.2816 × 1.400 = 45.21
Type II error: do not reject H_0, sample mean > 45.21, where mean ~ N(44, 1.400)

$z = \frac{45.21 - 44}{1.400} = 0.861$

Probability = 0.195

Answer: (a) sample mean < 45.2 (b) 0.195

10 Binomial expansion with partial fractions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{10x - 7}{(1 - x)(1 - 2x)}$.

(a) Express f(x) in partial fractions.

(b) Hence expand f(x) in ascending powers of x up to x^3, stating the range of validity.

(a) $10x - 7 \equiv A(1 - 2x) + B(1 - x)$: $A = -3$, $B = -4$.

$f(x) = \frac{-3}{1 - x} - \frac{4}{1 - 2x}$.

(b) $\frac{1}{1 + cx} = 1 - cx + c^2x^2 - c^3x^3 + \ldots$ (use $c = -1$, then $c = -2$).

Coefficients of x^j: $-3(1)^j - 4(2)^j$.

Valid for $|x| < \frac{1}{2}$ (both expansions).

Answer: (a) $A = -3$, $B = -4$ (b) $-7 - 11x - 19x^2 - 35x^3 + \ldots$; $|x| < \frac{1}{2}$