Day 349 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Ladder against a smooth wall: how high can a person climb (AQA, Edexcel, OCR, MEI)

A uniform ladder AB of length 4 m and weight 250 N has A on rough ground ($\mu = 0.5$) and B against a smooth vertical wall. It makes angle θ with the ground, $\tan\theta = \frac{4}{3}$.

A man of weight 500 N climbs the ladder. How far along the ladder can he go before it slips?

Wall reaction S horizontal, ground reaction R vertical, friction F = S. Moments about A: $S \times L \sin\theta = W \times \frac{L}{2}\cos\theta$

So $S = \frac{W}{2\tan\theta}$. Unloaded, R = W: need $\frac{W}{2\tan\theta} \le \mu W$, so $\mu \ge \frac{1}{2 \times \frac{4}{3}} = 0.375$

With the man at distance x: R = 750, and moments about A give $S\,L\sin\theta = (250 \times 2 + 500\,x)\cos\theta$

Limiting: $S = \mu R = 0.5 \times 750 = 375$, so $375 \times 4 \times \frac{4}{3} = 250 \times 2 + 500x$

$500x = 2000 - 500$, so $x = 3$

Answer: 3 m

2 Geometric series: least n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A geometric series has first term 6 and common ratio 5.

Find the least n for which $S_n > 62640$.

$S_n = \frac{6(5^n - 1)}{4} > 62640$.

$5^n > 41761$.

$n > \frac{\log(4.176e{+}04)}{\log(5)} = 6.611$.

So the least n is 7.

Answer: n = 7

3 Normal: unknown standard deviation then a second probability (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The times taken by a bus, in minutes are modelled by $N(39, \sigma^2)$. The probability of a value above 52 is 0.05.

(a) Find σ.

(b) Find the probability of a value below 51.

$P(X > 52) = 0.05$ so $Z = \frac{52 - 39}{\sigma} = 1.6449$

$\sigma = \frac{13}{1.6449} = 7.903$

$z = \frac{51 - 39}{7.903} = 1.518$

$P(X < 51) = P(Z < 1.518) = 0.936$

Answer: (a) 7.90 (b) 0.936

4 Binomial expansion (integer powers) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the term independent of x in the expansion of $(3x^2 + \frac{1}{x^2})^{10}$.

General term: $C(10,r)(3x^2)^{10-r}(\frac{1}{x^2})^r$.

Power of x: $2(10 - r) - 2r = 0$, so $r = 5$.

Term = $C(10,5) \times 3^5 \times (1)^5 = 61236$.

Answer: 61236

5 Binomial: mean and variance given (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A binomial random variable X has mean 16 and variance 9.6.

(a) Find n and p.

(b) Find P(X = 17), correct to 3 significant figures.

$np = 16$ and $np(1 - p) = 9.6$

Divide: $1 - p = 9.6 \div 16 = \frac{3}{5}$, so $p = \frac{2}{5}$

$n = 16 \div \frac{2}{5} = 40$

$P(X = 17) = C(40, 17)\ (\frac{2}{5})^{17}\ (\frac{3}{5})^{23} = 0.120$

Answer: (a) n = 40, p = $\frac{2}{5}$ (b) 0.120

6 Inverse trigonometric values (Edexcel, OCR, MEI, Eduqas, CIE (inverse trig))

Find the exact value of

$\cos(2\cos^{-1}(\frac{12}{13}))$.

Let $A = \cos^{-1}(\frac{12}{13})$, so $\cos A = \frac{12}{13}$

$\cos 2A = 2\cos^2 A - 1 = 2 \times \frac{144}{169} - 1$

Answer: $\frac{119}{169}$

7 Least force to drag a box (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of weight 175 N is pulled at constant speed along rough horizontal ground ($\mu = \frac{7}{24}$) by a force P N at angle θ above the horizontal.

(a) Find P when $\tan\theta = \frac{4}{3}$. (b) Find the least possible value of P as θ varies, and the corresponding θ to 1 d.p.

Vertical: $R + P\sin\theta = 175$. Horizontal: $P\cos\theta = \mu R = \frac{7}{24}(175 - P\sin\theta)$

$P(\cos\theta + \frac{7}{24}\sin\theta) = \frac{7}{24} \times 175$, so $P = \frac{1225}{24} \div (\cos\theta + \frac{7}{24}\sin\theta)$

(a) $\cos\theta = \frac{3}{5}$, $\sin\theta = \frac{4}{5}$: $P = \frac{1225}{24} \div \frac{5}{6} = 61.25$

(b) P is least when $\cos\theta + \frac{7}{24}\sin\theta$ is greatest. Its greatest value is $\sqrt{(1 + \frac{7}{24}^2)} = \frac{25}{24}$,

which occurs when $\tan\theta = \frac{7}{24}$. Then $P = \frac{1225}{24} \div \frac{25}{24} = 49$ and $\theta = 16.3°$

Answer: (a) 61.25 N (b) 49 N at θ = 16.3°

8 Addition formulae with exact values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A is acute and B is obtuse. $\cos A = \frac{15}{17}$ and $\sin B = \frac{3}{5}$.

Find the exact values of $\tan(A - B)$ and $\cos(A + B)$.

$\cos A = \frac{15}{17}$, $\sin A = \frac{8}{17}$; $\sin B = \frac{3}{5}$, $\cos B = -\frac{4}{5}$

$\tan(A - B) = \frac{\sin(A - B)}{\cos(A - B)} = \frac{-\frac{77}{85}}{-\frac{36}{85}} = \frac{77}{36}$

$\cos(A + B) = \cos A\cos B - \sin A\sin B = -\frac{84}{85}$

Answer: $\tan(A - B) = \frac{77}{36}$, $\cos(A + B) = -\frac{84}{85}$

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9 Inverse function and range (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The function f is defined by $f(x) = 2x^2 - 12x + 26$ for $x \leq k$. Given that k is the greatest value for which f is one-one, find k, $f^{-1}(x)$, and the domain of f^{-1}.

Complete the square: $f(x) = 2(x - 3)^2 + 8$.
The vertex is at $x = 3$, so $k = 3$ (f is one-one on one side of it).

Let $y = f(x)$: $y - 8 = 2(x - 3)^2$, so $(x - 3) = \pm\sqrt{\frac{y - 8}{2}}$.

As $x \leq 3$, take the negative root: $x = 3 - \sqrt{\frac{y - 8}{2}}$.

Range of f = domain of f^{-1}: $f(3) = 8$, so $x \geq 8$.

Answer: $k = 3$; $f^{-1}(x) = 3 - \sqrt{\frac{x - 8}{2}}$; $x \geq 8$

10 Area under a parametric curve (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = t^2$, $y = 3t(3 - t)$, for $0 \leq t \leq 3$. The curve meets the x-axis at O and at A. Find the area enclosed between the curve and the x-axis.

$dx/dt = 2t$; x from 0 to 9 as t goes from 0 to 3

Area $= \int_0^3 y \frac{dx}{dt}\, dt = \int_0^3 6t^2(3 - t)\, dt$

$= 6[\frac{3t^3}{3} - \frac{t^4}{4}]$

Answer: $\frac{81}{2}$