Day 346 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Quadratic sequence from recurrence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = u_n + n + 5$, $u_1 = -1$.

(a) Show that $u_n = an^2 + bn + c$ and find a, b, c.

(b) Find the least n for which $u_n > 1217$.

$u_n = u_1 + \sum_{k=1}^{n-1}(k + 5) = -1 + 1 \times \frac{(n-1)n}{2} + 5(n-1)$.

$= \frac{1}{2}n^2 + \frac{9}{2}n - 6$.

So $a = \frac{1}{2}$, $b = \frac{9}{2}$, $c = -6$.

$u_{45} = 1209 \leq 1217$ and $u_{46} = 1259 > 1217$.

Answer: (a) $a = \frac{1}{2}$, $b = \frac{9}{2}$, $c = -6$ (b) $n = 46$

2 Replacing an observation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

12 recorded times (seconds) have mean 54 and standard deviation 4.
One value was recorded as 49 but should have been 41.
Find the correct mean and the correct standard deviation.

$\Sigma x = 12 \times 54 = 648$, so corrected $\Sigma x = 648 - 49 + 41 = 640$

Correct mean $= \frac{640}{12} = 53.33$

$\Sigma x^2 = 12(16 + 2916) = 35184$, corrected $\Sigma x^2 = 35184 - 2401 + 1681 = 34464$

Variance $= \frac{34464}{12} - 53.333^2 = 27.556$, $\sigma = 5.249$

Answer: mean 53.3, standard deviation 5.25

3 Overtaking with a delayed start (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Car A passes a point O travelling at a constant speed of 12 m s^{-1}. 5 s later, car B leaves O from rest with constant acceleration 1.5 m s^{-2}, travelling in the same direction.
B overtakes A at the point X. Find (a) the time B takes to reach X, (b) the distance OX, (c) the speed of B at X.

Let B travel for t s. A: distance = 12(t + 5); B: distance = ½ × 1.5 × t^2

Equate: $1.5t^2 - 24t - 120 = 0$, so $(t - 20)(1.5t + 6) = 0$

t = 20 (the other root is negative)

OX = $1.5 \times 20^2 / 2 = 300$ m, and speed = $1.5 \times 20 = 30$

Answer: (a) 20 s (b) 300 m (c) 30 m s^{-1}

4 Equations quadratic in an exponential (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $3^{2x} - 11 \times 3^x + 10 = 0$,
giving each answer exactly and, where not an integer, to 3 s.f.

Let $u = 3^x$: $u^2 - 11u + 10 = 0$

$(u - 1)(u - 10) = 0$, so $3^x = 1$ or $3^x = 10$

Take logarithms: $x = \log_3 1 = 0$; $x = \log_3 10 = 2.10$

Answer: $x = \log_3 1 = 0$; $x = \log_3 10 = 2.10$

5 Reciprocal graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the curve C, its asymptotes (dashed) and a point P. C has equation $y = \frac{a}{x + p} + q$. Find a, p and q.

Asymptotes $x = -1$ and $y = 3$, so $p = 1$ and $q = 3$.

$y = \frac{a}{x + 1} + 3$.

P(−2, 6): $6 = \frac{a}{-1} + 3$, so $a = -3$.

Answer: $a = -3$, $p = 1$, $q = 3$

6 Inverse trigonometric values (Edexcel, OCR, MEI, Eduqas, CIE (inverse trig))

Find the exact value of $\tan(2\cos^{-1}(\frac{15}{17}))$.

Let $A = \cos^{-1}(\frac{15}{17})$: $\cos A = \frac{15}{17}$, $\sin A = \frac{8}{17}$, $\tan A = \frac{8}{15}$

$\tan 2A = \frac{2\tan A}{1 - \tan^2 A} = \frac{\frac{16}{15}}{\frac{161}{225}}$

Answer: $\frac{240}{161}$

7 Screening test and Bayes (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A rare allele is present in 1% of a population. A test is positive for 95% of those with it and for 1% of those without it. Find P(positive) and P(has it | positive).

P(positive) = 0.01 × 0.95 + 0.99 × 0.01
= 0.0095 + 0.0099 = 0.0194

P(has it | positive) = $\frac{0.0095}{0.0194}$

= 0.4897

Answer: (a) 0.0194 (b) 0.490

8 Transforming a graph: images of points (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows y = f(x), made of three straight line segments. Find the coordinates of the images of A, B, C and D on the graph of $y = 2f(x - 1) + 2$.

Inside f: replace x by $x - 1$, so x-coordinates become $(x + 1) \div 1$.
Outside: y-coordinates become $2 \times y + 2$.
A: (−3, −4) B: (1, 8)
C: (5, 2) D: (7, 12)
Answer: A′(−3, −4), B′(1, 8), C′(5, 2), D′(7, 12)

9 Projectile through a given point: two angles (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground at 35 m s^{-1}, at angle θ above the horizontal. It passes through the point P, 72 m horizontally and 40 m vertically from O.

Find the two possible values of tan θ, and the two possible values of θ to 1 d.p. Take g = 9.8 m s^{-2}.

Horizontal: $x = 35 \cos\theta \times t$. Vertical: $y = 35 \sin\theta \times t - 4.9t^2$

Eliminating t: $y = x \tan\theta - \frac{9.8x^2}{2u^2}(1 + \tan^2\theta)$

$\frac{9.8 \times 72^2}{2 \times 35^2} = 20.736$, so $40 = 72 \tan\theta - 20.736(1 + \tan^2\theta)$

$20.736 \tan^2\theta - 72 \tan\theta + 60.736 = 0$

$\tan\theta = \frac{13}{9}$ or $\frac{73}{36}$, so $\theta = 55.3°$ or $63.7°$

Answer: $\tan\theta = \frac{13}{9}$ or $\frac{73}{36}$; $\theta = 55.3°$ or $63.7°$

10 Exponential times trigonometric (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$y = e^{3x} \cos x$. Show that $dy/dx = e^{3x}(3 \cos x - \sin x)$.

Find the x coordinate of the stationary point with the smallest positive x (3 s.f.), and its nature.

Product rule: $dy/dx = (3)e^{3x} \cos x - e^{3x} \times \sin x$

$dy/dx = 0$ requires $3 \cos x = 1 \sin x$

$\tan x = 3$, smallest positive $x = 1.2490$

$x = 1.25$

Second derivative test (or gradient change): a maximum

Answer: x = 1.25, maximum