Day 345 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Sectors: perimeter and area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sector of a circle has radius r cm and angle θ radians, where $\theta < 2\pi$.
Its perimeter is 52 cm and its area is 168 cm^2.
Find the two possible pairs of values of r and θ.

Arc = $r\theta$, so $2r + r\theta = 52$ and $\frac{1}{2}r^2\theta = 168$

$r\theta = 52 - 2r$, so $\frac{1}{2}r(52 - 2r) = 168$

$r^2 - 26r + 168 = 0$, so $(r - 12)(r - 14) = 0$

r = 12: $r\theta = 28$, $\theta = \frac{7}{3}$ r = 14: $r\theta = 24$, $\theta = \frac{12}{7}$

Answer: r = 12, $\theta = \frac{7}{3}$ or r = 14, $\theta = \frac{12}{7}$

2 Recurrence: limit and convergence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = \frac{1}{3}u_n + 8$, $u_1 = -4$.

(a) Find the limit L of the sequence.
(b) Find the least n such that $|u_n - L| < 0.001$.

(a) At the limit $L = \frac{1}{3}L + 8$, so $L(1 - \frac{1}{3}) = 8$.

L = 12.

(b) $u_n - L = (u_1 - L)(\frac{1}{3})^{n-1} = (-16)(\frac{1}{3})^{n-1}$.

Need $16 \times (\frac{1}{3})^{n-1} < 0.001$, so $n - 1 > 8.811$.

The least n is 10.

Answer: (a) L = 12 (b) n = 10

3 Newton-Raphson iteration (AQA, Edexcel, OCR, MEI, Eduqas (Newton-Raphson))

$f(x) = e^x + 2x - 11$.
(a) Show that $f(x) = 0$ has a root α with $1 < \alpha < 2$.
(b) Use Newton-Raphson with $x_0 = 2$ to find x_1 and x_2
to 4 decimal places.

f(1) = −6.28 and f(2) = 0.389: a change of sign, f continuous

$f'(x) = e^x + 2$, so $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x_1 = 2 - \frac{0.3891}{9.389} = 1.9586$

$x_2 = 1.9586 - \frac{0.006257}{9.089} = 1.9579$

Answer: $x_1 = 1.9586$, $x_2 = 1.9579$

4 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

A shell is projected from a point O on horizontal ground with speed 63.7 m s^{-1} at an angle θ above the horizontal, where $\tan\theta = \frac{12}{5}$. Find (a) the greatest height, (b) the speed when it is at half of this height (3 s.f.). Take g = 9.8 m s^{-2}.

Vertical: $0 = 58.8^2 - 2 \times 9.8 \times H$, so H = 176.4

At height $\frac{H}{2}$ = 88.2: $v_y^2 = 58.8^2 - 2 \times 9.8 \times 88.2 = 1728.72$

v_x = 24.5 (constant), so $speed^2 = 600.25 + 1728.72 = 2328.97$

Answer: (a) 176.4 m (b) 48.3 m s^{-1}

5 Fixed point iteration and rearrangement (AQA, Edexcel, OCR, MEI, Eduqas, CIE (iterative methods))

The equation $e^x = 5x + 8$ has a root α.
(a) Show that the equation can be written as $x = \ln(5x + 8)$.
(b) Use $x_{n+1} = \ln(5x_n + 8)$ with $x_0 = 3$ to find x_1, x_2, x_3 to 4 d.p., and hence state α to 3 d.p.

Take natural logarithms of both sides to obtain $x = \ln(5x + 8)$
$x_1 = 3.1355$, $x_2 = 3.1645$, $x_3 = 3.1706$
The iteration converges (successive values agree to 3 d.p.): $\alpha = 3.172$
Answer: $x_1 = 3.1355$, $x_2 = 3.1645$, $x_3 = 3.1706$; $\alpha = 3.172$

6 Normal approximation to a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(150, 0.2)$.
(a) State why a normal approximation is suitable for X.
(b) Use a normal approximation to estimate $P(X \geq 40)$, to 3 significant figures.

$np = 30$ and $nq = 120$, both greater than 5 (n large, p not extreme)
$Y \sim N(30, 24)$ with $\sigma = 4.899$
Apply the continuity correction:
$P(X \geq 40) \approx P(Y > 39.5) = P(Z > 1.939) = 0.0262$
Answer: 0.0262

7 Composite functions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = -x - 1$ and $g(x) = x^2 - 3$.
Solve $fg(x) = gf(x)$.

$fg(x) = -(x^2 - 3) - 1 = -x^2 + 2$.
$gf(x) = (-x - 1)^2 - 3 = x^2 + 2x - 2$.
Equate: $2x^2 + 2x - 4 = 0$.
Solve the quadratic: $x = -2$ or $x = 1$.
Answer: $x = -2$ or $x = 1$

8 Non-uniform rod: centre of mass and tilting (AQA, Edexcel, OCR, MEI)

A non-uniform rod AB of length 5 m and mass 8 kg rests on supports at C and D, where AC = 2 m and AD = 4 m, with $R_C = 3 R_D$.
Find (a) the distance of the centre of mass from A. A particle of mass m kg at B puts the rod on the point of tilting about D.
(b) Find m.

$R_C = 3 R_D$. Moments about A: $R_C \times 2 + R_D \times 4 = 8\,g \times AG$
$R_D(3 \times 2 + 4) = 8\,g \times AG$ with $R_D(1 + 3) = 8\,g$, so $AG = \frac{3 \times 2 + 4}{4} = 2.5$
Tilting about D: $R_C = 0$. Moments about D: $m\,g\,(5 - 4) = 8\,g\,(4 - 2.5)$
$m = \frac{8 \times 1.5}{1} = 12$
Answer: (a) 2.5 m (b) 12

9 Reciprocal graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the curve C, its asymptotes (dashed) and a point P. C has equation $y = \frac{a}{x + p} + q$. Find a, p and q.

Asymptotes $x = 3$ and $y = -2$, so $p = -3$ and $q = -2$.
$y = \frac{a}{x - 3} - 2$.
P(1, −4): $-4 = \frac{a}{-2} - 2$, so $a = 4$.
Answer: $a = 4$, $p = -3$, $q = -2$

10 Tree diagram with an unknown probability (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The tree diagram shows two games played in turn.

Given the second is a H, P(first was a H) = $\frac{1}{3}$. Find x.

P(H then H) = 0.2x, P(second H) = 0.2x + 0.6(1 − x)

P(first H | second H) = $\frac{0.2x}{0.2x + 0.6(1 - x)} = \frac{1}{3}$

$0.2x = \frac{1}{3}(0.2x + 0.6 - 0.6x)$

Solve the linear equation for x: $x = \frac{3}{5}$

Answer: x = $\frac{3}{5}$