Day 343 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Distribution of the larger of two scores (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Two fair 8-sided dice, numbered 1 to 8, are rolled. M is the smaller of the two scores.

(a) Show that $P(M = m) = \frac{(17 - 2m)}{64}$ for m = 1, 2, ..., 8.

(b) Find P(M = 3) and E(M).

(a) $M \geq m$ means both scores $\geq m$: $P(M \geq m) = (\frac{8 - m + 1}{8})^2$, so $P(M = m) = \frac{(8 - m + 1)^2 - (8 - m)^2}{64} = \frac{(17 - 2m)}{64}$

(b) $P(M = 3) = \frac{11}{64}$

$E(M) = \Sigma\, m\, P(M = m) = \frac{1}{64} \Sigma\, m(17 - 2m)$

$= \frac{51}{16}$

Answer: (b) $P(M = 3) = \frac{11}{64}$, $E(M) = \frac{51}{16}$

2 Double angle equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\cos 2\theta - \cos \theta = 0$
for $0 \leq \theta < 360°$.

Use $\cos 2\theta \equiv 2\cos^2\theta - 1$: $2\cos^2\theta - \cos \theta - 1 = 0$
$(2 \cos \theta + 1)(\cos \theta - 1) = 0$

$\cos \theta = -\frac{1}{2}$ or $\cos \theta = 1$

Using the CAST diagram for $0 \leq \theta < 360°$

Answer: θ = 0°, 120°, 240°

3 Surds: rationalising the denominator (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{6 + 3\sqrt{7}}{3 + \sqrt{7}}$ in the form $p + q\sqrt{7}$, where p and q are rational numbers.

Multiply top and bottom by the conjugate $(3 - \sqrt{7})$.
Numerator: $(6 + 3\sqrt{7})(3 - \sqrt{7}) = -3 + 3\sqrt{7}$.
Denominator: $3^2 - 1^2(7) = 2$.

So the fraction = $\frac{-3 + 3\sqrt{7}}{2}$.

Answer: $-\frac{3}{2} + \frac{3}{2}\sqrt{7}$

4 Particle on a rough table, string slack when B lands (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particles A (3 kg) and B (3 kg) are joined by a light string over a smooth pulley (see diagram). A lies on a rough table with $\mu = 0.5$, and B is 0.9 m above the floor.
The system is released from rest. Find the total distance A moves before coming to rest (A does not reach the pulley).

Until B lands, A: $T - 0.5 \times 3 \times 9.8 = 3a$. B: $3 \times 9.8 - T = 3a$
Adding: $(3 - 0.5 \times 3) \times 9.8 = 6a$, so $a = 2.45$
$v^2 = 2 \times 2.45 \times 0.9 = 4.41$, so $v = 2.1$

After B lands, A decelerates at $\mu g = 4.9$: extra distance = $\frac{2.1^2}{2 \times 4.9} = 0.45$

Total distance = 0.9 + 0.45

Answer: 1.35 m

5 Newton cooling model (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A drink cools according to $\theta = 20 + Ae^{-kt}$, where θ °C is the temperature after t minutes. Initially $\theta = 95$, and after 6 minutes $\theta = 70$.
Find A and k, then find when $\theta = 47$ (3 s.f.).

$t = 0$: $20 + A = 95$ so $A = 75$

$t = 6$: $20 + 75e^{-6k} = 70$ so $e^{-6k} = \frac{2}{3}$

$k = \frac{1}{6} \ln \frac{3}{2} = 0.0676$

$75\, e^{-kt} = 27$ so $t = \frac{1}{k} \ln \frac{25}{9}$

$t = 15.1$ minutes

Answer: A = 75, k = 0.0676, t = 15.1 minutes

6 Screening test and Bayes (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A rare allele is present in 1% of a population. A test is positive for 90% of those with it and for 8% of those without it.
Find P(positive) and P(has it | positive).

$P(\text{positive}) = 0.01 \times 0.9 + 0.99 \times 0.08$
$= 0.009 + 0.0792 = 0.0882$

$P(\text{has it} \mid \text{positive}) = \frac{0.009}{0.0882}$

$= 0.1020$

Answer: (a) 0.0882 (b) 0.102

7 Partial fractions (improper) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{3x^3 - 19x^2 + 40x - 31}{(x-2)(x-3)}$ in the form $ax + b + \frac{C}{(x-2)} + \frac{D}{(x-3)}$.

Divide: the numerator has degree 3, the denominator degree 2.
$(3x^3 - 19x^2 + 40x - 31) \div (x^2 - 5x + 6)$ gives quotient $3x - 4$.
Remainder: $2x - 7$.
Put $x = 2$: remainder $= C(-1)$ so $C = 3$. Put $x = 3$: $D = -1$.

Answer: $3x - 4 + \frac{3}{(x-2)} - \frac{1}{(x-3)}$

8 Quadratic sequence from recurrence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = u_n + n - 1$, $u_1 = -1$.
(a) Show that $u_n = an^2 + bn + c$ and find a, b, c.
(b) Find the least n for which $u_n > 506$.

$u_n = u_1 + \sum_{k=1}^{n-1} (k - 1) = -1 + 1 \times \frac{(n-1)n}{2} - 1(n-1)$.

$= \frac{1}{2}n^2 - \frac{3}{2}n$.

So $a = \frac{1}{2}$, $b = -\frac{3}{2}$, $c = 0$.

$u_{33} = 495 \leq 506$ and $u_{34} = 527 > 506$.

Answer: (a) $a = \frac{1}{2}$, $b = -\frac{3}{2}$, $c = 0$ (b) $n = 34$

9 Block on a plank: limiting slip (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A plank of mass 2 kg lies on a smooth horizontal table, with a block of mass 2 kg on it ($\mu = 0.3$ between them). A horizontal force P N acts on the plank.

(a) Find the greatest P for which the block does not slip. (b) Find the accelerations of block and plank when P = 35.28. Take $g = 9.8$ m s^{-2}.

No slipping: common acceleration $a = \frac{P}{4}$. The block is accelerated only by friction: $F = 2a \leq \mu mg$

So $a \leq \mu g = 2.94$, and $P_{max} = 4 \times 2.94 = 11.76$

$P = 35.28 > 11.76$, so the block slips. Block: $a = \mu g = 2.94$

Plank: $35.28 - 5.88 = 2a$, so $a = 14.7$

Answer: (a) 11.76 N (b) block 2.94 m s^{-2}, plank 14.7 m s^{-2}

10 Lines in 3D: intersection and angle (CIE (9709); Further Maths on other boards)

Line l_1: $\mathbf{r} = \begin{pmatrix} 7 \\ 4 \\ -3 \end{pmatrix} + s\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$,

line l_2: $\mathbf{r} = \begin{pmatrix} -2 \\ 8 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}$.

Show whether the lines meet, giving the point of intersection if they do, and find the acute angle between them to 1 d.p.

Equate components: $7 + 3s = -2 + 2t$; $4 + 2s = 8 - 2t$; $-3 = 3 - 2t$

Solving the first two: $s = -1$, $t = 3$; the third equation is satisfied

Intersection: $(4, 2, -3)$

$\cos\varphi = \frac{|2|}{\sqrt{13}\sqrt{12}} = 0.1601$, $\varphi = 80.8°$

Answer: Lines meet at $(4, 2, -3)$; angle 80.8°