Day 340 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Double angle equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\cos 2\theta - 3\sin\theta - 2 = 0$
for $0 \leq \theta < 2\pi$, giving answers in terms of π.

Use $\cos 2\theta \equiv 1 - 2\sin^2\theta$: $2\sin^2\theta + 3\sin\theta + 1 = 0$
$(2\sin\theta + 1)(\sin\theta + 1) = 0$

$\sin\theta = -\frac{1}{2}$ or $\sin\theta = -1$

Principal values and symmetry in $[0, 2\pi)$

Answer: $\theta = \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

2 Just clearing a wall at 45 degrees (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground at 45° to the horizontal. The ball just passes over the top of a vertical wall of height 4.8 m, at a horizontal distance of 93 m from O.
Find (a) the speed of projection, (b) the greatest height of the ball above the ground, to 3 s.f. Take $g = 9.8$ m s^{-2}.

At 45°: $x = \frac{u\,t}{\sqrt{2}}$ and $y = \frac{u\,t}{\sqrt{2}} - 4.9t^2$, so $y = x - \frac{9.8x^2}{u^2}$

$4.8 = 93 - \frac{9.8 \times 93^2}{u^2}$, so $u^2 = \frac{9.8 \times 8649}{88.2} = 961$

$u = 31$

Greatest height $= \frac{(u \sin 45°)^2}{(2 \times 9.8)} = \frac{961}{39.2}$

Answer: (a) 31 m s^{-1} (b) 24.5 m

3 R-form: solving (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Express $20\cos\theta - 21\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < 90°$. Give α to 1 d.p.
(b) Hence solve $20\cos\theta - 21\sin\theta = 23$ for $0 \leq \theta < 360°$, giving answers to 1 d.p.

$R = \sqrt{20^2 + 21^2} = 29$ and $\tan\alpha = \frac{21}{20}$, so $\alpha = 46.4°$

So $20\cos\theta - 21\sin\theta \equiv 29\cos(\theta + 46.4°)$

$\cos(\theta + \alpha) = \frac{23}{29}$

$\theta + \alpha = \pm 37.5°$ (using $\alpha = 46.4°$)
Adding or subtracting multiples of 360° to give $0 \leq \theta < 360°$
Answer: (a) $R = 29$, $\alpha = 46.4°$ (b) $\theta = 276.1°, 351.1°$

4 Cubic with only one real root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 + 3x^2 - 3x + 4$.
Show that $(x + 4)$ is a factor of $f(x)$, and that the equation $f(x) = 0$ has exactly one real root.

$f(-4) = -64 + 48 + 12 + 4 = 0$, so $(x + 4)$ is a factor.

$f(x) = (x + 4)(x^2 - x + 1)$.

For $x^2 - x + 1 = 0$: $b^2 - 4ac = 1 - 4 = -3 < 0$.
No real roots from the quadratic, so the only real root is $x = -4$.
Answer: Proved; the only real root is $x = -4$

5 Direction of motion at 45 degrees (AQA, Edexcel, OCR, MEI, Eduqas)

A shell is projected from level ground with horizontal and vertical velocity components 14.7 and 53.9 m s^{-1}.
Find (a) the two times at which it moves at 45° to the horizontal, (b) the horizontal distance between those two points.
Take $g = 9.8$ m s^{-2}.

At 45° to the horizontal, $|v_y| = v_x = 14.7$
Moving upwards: $53.9 - 9.8t = 14.7$, so $t = 4$
Moving downwards: $53.9 - 9.8t = -14.7$, so $t = 7$
Horizontal distance $= 14.7 \times (7 - 4)$
Answer: (a) 4 s and 7 s (b) 44.1 m

6 Parametric curve and line intersection (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has parametric equations
$x = 2t$, $y = 2t^2$. The line $y = 5x - 8$
meets C at A and B. Find the exact length AB.

Substitute: $2t^2 = 10t - 8$.
$2t^2 - 10t + 8 = 0 \Rightarrow t = 4$ or $t = 1$.
A(8, 32), B(2, 2).
$AB^2 = 6^2 + 30^2 = 936$.
Answer: AB = $6\sqrt{26}$

7 Most likely value of a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(22, \frac{3}{10})$.

(a) Show that $P(X = r + 1) \div P(X = r) = \frac{22 - r}{r + 1} \times \frac{3}{7}$.

(b) Find the most likely value of X and its probability (3 s.f.).

$P(X = r + 1) \div P(X = r) = \frac{C(22, r + 1)}{C(22, r)} \times \frac{p}{q} = \frac{22 - r}{r + 1} \times \frac{p}{q}$

With $p = \frac{3}{10}$ and $q = \frac{7}{10}$, $p \div q = \frac{3}{7}$

The ratio exceeds 1 when $(22 - r)\frac{3}{10} > (r + 1)\frac{7}{10}$, that is $r < 5.9$

So probabilities rise up to r = 6 and then fall; the mode is 6
P(X = 6) = 0.181
Answer: 6 with probability 0.181

8 Triangle in 3D: angle and area (CIE (9709); Further Maths on other boards)

The points A, B, C have coordinates (5, −1, 5), (1, −1, 4), (−2, 1, 3).
Find angle ABC to 1 d.p., and the exact area of triangle ABC.

$BA = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$, $BC = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$; $BA \cdot BC = -13$

$|BA| = \sqrt{17}$, $|BC| = \sqrt{14}$

$\cos ABC = \frac{-13}{\sqrt{17}\sqrt{14}} = -0.8427$, so ABC = 147.4°

$|BA \times BC| = \left|\begin{pmatrix} -2 \\ 1 \\ 8 \end{pmatrix}\right| = \sqrt{69}$

Area $= \frac{1}{2}|BA \times BC|$

Answer: ABC = 147.4°, area = $\frac{1}{2}\sqrt{69}$

9 Circle touching both axes (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A circle has its centre in the first quadrant and touches both the x-axis and the y-axis. It passes through the point (9, 8).
Find the two possible equations of the circle.

Centre (r, r), radius r: $(x - r)^2 + (y - r)^2 = r^2$.
Substitute (9, 8): $(9 - r)^2 + (8 - r)^2 = r^2$.
$r^2 - 34r + 145 = 0$.
$(r - 5)(r - 29) = 0$, so r = 5 or r = 29.
Answer: $(x - 5)^2 + (y - 5)^2 = 25$ or $(x - 29)^2 + (y - 29)^2 = 841$

10 Binomial: mean and variance given (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A binomial random variable X has mean 3 and variance 2.4.

(a) Find n and p.

(b) Find P(X = 5), correct to 3 significant figures.

$np = 3$ and $np(1 - p) = 2.4$

Divide: $1 - p = 2.4 \div 3 = \frac{4}{5}$, so $p = \frac{1}{5}$

$n = 3 \div \frac{1}{5} = 15$

$P(X = 5) = C(15, 5) (\frac{1}{5})^5 (\frac{4}{5})^{10} = 0.103$

Answer: (a) n = 15, p = $\frac{1}{5}$ (b) 0.103