Day 34 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Related rates of change (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A cone with its vertex downwards and semi-vertical angle α, where $\tan\alpha = \frac{1}{2}$, is filled at 3π cm^{3}s^{-1}. Find the rate at which the depth h is rising when h = 5 cm. ($V = \frac{1}{3}\pi r^2 h$)

$r = \frac{h}{2}$, so $V = \frac{1}{3}\pi \times \frac{h^2}{4} \times h = \frac{\pi}{12}h^3$

$dV/dt = \frac{\pi}{4}h^2\, dh/dt$, so $3\pi = \frac{\pi}{4} \times 25 \times dh/dt$

$dh/dt = \frac{12}{25}$

Answer: $\frac{12}{25}$ cm s^{-1}

2 Replacing an observation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

25 recorded masses (grams) have mean 47 and standard deviation 8.
One value was recorded as 67 but should have been 79.
Find the correct mean and the correct standard deviation.

$\Sigma x = 25 \times 47 = 1175$, so corrected $\Sigma x = 1175 - 67 + 79 = 1187$

Correct mean $= \frac{1187}{25} = 47.48$

$\Sigma x^2 = 25(64 + 2209) = 56825$, corrected $\Sigma x^2 = 56825 - 4489 + 6241 = 58577$

Variance $= \frac{58577}{25} - 47.480^2 = 88.730$, $\sigma = 9.420$

Answer: mean 47.5, standard deviation 9.42

3 Velocity-time graph with a return (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the velocity-time graph of a particle P moving on a straight line (velocity in m s^{-1}, time in s). At time 15 s, P is back at its starting point.
Find (a) the value of W, (b) the greatest distance of P from its starting point.

Net area = 0 (back at start): ½ × 3 × 4 + ½(4 − W)(6) − W(6) = 0
6 + 12 − 3W − 6W = 0, so 9W = 18
W = 2

v = 0 where the line from (3, 4) to (9, −2) crosses the axis: $t = 3 + 6 \times \frac{4}{6} = 7$

Greatest distance = area of triangle = ½ × 7 × 4

Answer: (a) W = 2 (b) 14 m

4 Segments: exact area and perimeter (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A chord AB of a circle, centre O and radius 10 cm, subtends an angle of $\frac{2\pi}{3}$ at O.

Find the exact area and the exact perimeter of the minor segment cut off by AB.

Sector OAB: $\frac{1}{2}r^2\theta = \frac{1}{2} \times 10^2 \times \frac{2\pi}{3}$; triangle OAB: $\frac{1}{2}r^2\sin\theta = 25\sqrt{3}$

Area $= \frac{100}{3}\pi - 25\sqrt{3}$

Chord AB $= 2r\sin(\theta/2) = 10\sqrt{3}$

Arc AB $= r\theta = \frac{20}{3}\pi$

Perimeter = arc + chord

Answer: Area = $\frac{100}{3}\pi - 25\sqrt{3}$ cm^2, perimeter = $\frac{20}{3}\pi + 10\sqrt{3}$ cm

5 Transformations of y = x^2 (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Describe a sequence of two transformations that maps the curve $y = x^2$ onto the curve $y = 3x^2 - 24x + 52$.

Complete the square: $y = 3(x - 4)^2 + 4$.

Step 1: $y = x^2 \rightarrow y = 3x^2$: a stretch, scale factor 3, parallel to the y-axis.

Step 2: replace x by $(x - 4)$ and add 4: translation by $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$.

Answer: a stretch, scale factor 3, parallel to the y-axis, then translation $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$

6 Poisson approximation to a binomial (CIE)

$X \sim B(100, 0.015)$.

(a) Explain why a Poisson approximation is suitable and state its distribution.

(b) Use it to estimate $P(X > 2)$, to 4 s.f.

(c) Find the exact value of $P(X > 2)$, to 4 s.f.

n = 100 is large and p = 0.015 is small, with np = 1.5

X is approximately Po(1.5)

$P(X > 2) \approx 1 - P(Y \leq 2) = 0.1912$

Exact: $1 - P(X \leq 2)$ with $X \sim B(100, 0.015) = 0.1902$

Answer: (a) Po(1.5) (b) 0.1912 (c) 0.1902

7 Velocity with a square-root term (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves on a straight line. At time t seconds, where $t \geq 0$, its velocity is v m s^{-1}, where $v = 4\sqrt{t} - 1t$.

Find (a) the greatest speed of P, (b) the greatest displacement of P from its starting point.

$\frac{dv}{dt} = \frac{4}{2\sqrt{t}} - 1 = 0$ when $\sqrt{t} = 2$, so t = 4

$v = 4 \times 2 - 1 \times 4 = 4$

$v = 0$ when $\sqrt{t}(4 - 1\sqrt{t}) = 0$: t = 16 (t = 0 is the start)

$s = \int v\, dt = \frac{2}{3} \times 4\, t^{\frac{3}{2}} - \frac{1}{2}t^2$, so at t = 16: $s = \frac{512}{3} - 128$

Answer: (a) 4 m s^{-1} (b) $\frac{128}{3}$ m

8 Equations in disguise (substitution) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $2x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 24 = 0$.

Let $u = x^{\frac{1}{3}}$: $2u^2 - 2u - 24 = 0$.

$(u - 4)(u + 3) = 0$, so u = 4 or u = −3.

Cube: x = 64 or x = −27.

Answer: x = −27 or x = 64

9 Points of inflection (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact coordinates of the point of inflection of $y = (x + 2)e^{-2x}$, justifying that it is a point of inflection.

$dy/dx = e^{-2x}(1 - 2(x + 2))$

$d^2y/dx^2 = 2e^{-2x}(2(x + 2) - 2) = 0$, so x = −1

d^2y/dx^2 changes sign either side of x = −1

$y = e^2$

Answer: $(-1, e^2)$

10 Tangent line to circle: finding k (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The line $4y = 3x + k$ is a tangent to the circle
$(x - 1)^2 + (y + 3)^2 = 1$.
Find the two possible values of k.

Substitute $y = \frac{3x + k}{4}$: $(x - 1)^2 + (\frac{3x + k}{4} + 3)^2 = 1$.

Multiply by 16 and collect: a quadratic in x with a = 25.

Tangent ⇒ discriminant = 0. This simplifies to $(k + 15)^2 = 25$.

$k + 15 = \pm 5$.

Answer: $k = -20$ or $k = -10$