Day 339 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Histogram: frequency density (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The histogram shows the times, t seconds, of some runners.
Find (a) the total number of runners, (b) an estimate of the number with $15 \leq t < 39$.

Frequency = frequency density × width. Bar 10 to 22: height 4 squares, width 12, frequency 48
So one square of frequency density = 1 runners per second
Class frequencies = 10, 48, 24, 6, 48
Total = 136
Estimate = $1 \times [4(7) + 3(8) + 1(6) + 4(3)] = 70$
Answer: (a) 136 (b) 70

2 Geometric series: least n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A geometric series has first term 4 and common ratio $\frac{9}{10}$.
Find the least n for which S_n exceeds 90% of the sum to infinity.

$S_n = S_\infty(1 - r^n) > 0.90\, S_\infty$.
$1 - \left(\frac{9}{10}\right)^n > 0.90$, so $\left(\frac{9}{10}\right)^n < 0.1$.
$n > \frac{\log(0.1)}{\log(0.9)} = 21.854$.
So the least n is 22.
Answer: n = 22

3 Compound change with logarithms (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A drug dose of 80 mg decays by 10% each hour.
Find the first whole number of hours after which less than 5 mg remains.

$80 \times 0.90^n < 5$, so $0.90^n < \frac{1}{16}$
$n > \frac{\ln \frac{1}{16}}{\ln 0.90} = 26.32$ (ln 0.90 is negative, so the inequality reverses)
n = 27
Answer: 27

4 Self-inverse function (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{kx + 13}{x - 5}$ for $x \neq 5$. Given that f is self-inverse, find k, and hence solve f(x) = x.

Let $y = f(x)$: $y(x - 5) = kx + 13$, so $x(y - k) = 5y + 13$.
$f^{-1}(x) = \frac{5x + 13}{x - k}$.
$f = f^{-1}$ requires $k = 5$.
$f(x) = x$: $5x + 13 = x(x - 5)$, so $x^2 - 10x - 13 = 0$.
$x = 5 \pm \sqrt{38} = 5 \pm \sqrt{38}$.
Answer: k = 5; x = $5 \pm \sqrt{38}$

5 Velocity parallel to a given direction (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves in a plane. At time t seconds its position vector relative to a fixed origin O is r metres, where $\mathbf{r} = (-1.5t^2 + 6t)\mathbf{i} + (-1.5t^2 - 6t)\mathbf{j}$.
Find (a) the value of t at which P is moving parallel to $(\mathbf{i} + 2\mathbf{j})$, (b) the speed of P at this time (3 s.f. if not exact).

$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-3t + 6)\mathbf{i} + (-3t - 6)\mathbf{j}$
Parallel to $(\mathbf{i} + 2\mathbf{j})$: $(-3t + 6) \times 2 = (-3t - 6) \times 1$
$t(-3) = -18$, so $t = 6$
At $t = 6$: $\mathbf{v} = (-12\mathbf{i} - 24\mathbf{j})$, speed $= \sqrt{(-12^2 + -24^2)}$
Answer: (a) t = 6 (b) 26.8 m s^{-1}

6 Derivative of the inverse function (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve is given by $x = y^3 - y^2 + y$.
Find dy/dx and d^2y/dx^2 when $y = -1$.

$dx/dy = 3y^2 - 2y + 1 = 6$ when $y = -1$, so $dy/dx = \frac{1}{dx/dy} = \frac{1}{6}$

$d^2x/dy^2 = -8$ when $y = -1$

$d^2y/dx^2 = -\frac{d^2x}{dy^2/(dx/dy)^3} = -\frac{-8}{6^3}$

Answer: $dy/dx = \frac{1}{6}$, $d^2y/dx^2 = \frac{1}{27}$

7 Inverse function and range (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The function f is defined by $f(x) = x^2 - 6x + 10$
for $x \geq k$. Given that k is the least value for which
f is one-one, find k, $f^{-1}(x)$, and the domain of f^{-1}.

Complete the square: $f(x) = (x - 3)^2 + 1$.
The vertex is at $x = 3$, so $k = 3$ (f is one-one on one side of it).
Let $y = f(x)$: $y - 1 = (x - 3)^2$, so $(x - 3) = \pm\sqrt{y - 1}$.
As $x \geq 3$, take the positive root: $x = 3 + \sqrt{y - 1}$.
Range of f = domain of f^{-1}: $f(3) = 1$, so $x \geq 1$.
Answer: $k = 3$; $f^{-1}(x) = 3 + \sqrt{x - 1}$; $x \geq 1$

8 Triangular pdf from a diagram (CIE)

The diagram shows the pdf of X.
Find (a) k, (b) E(X), (c) the median of X (3 s.f.).

(a) Area of triangle = ½ × 8 × k = 1, so $k = \frac{1}{4}$

(b) The mean of a triangular distribution is $\frac{0 + 7 + 8}{3} = 5$

$F(7) = \frac{7}{8} = \frac{7}{8} \geq \frac{1}{2}$, so the median is in the left part

$F(x) = \frac{x^2}{56} = \frac{1}{2}$, so $x = \sqrt{28} = 5.292$

Answer: (a) $k = \frac{1}{4}$ (b) E(X) = 5 (c) 5.29

9 Horizontal force on a rough slope: range of values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle of weight 125 N rests on a rough plane inclined at α to the horizontal, where $\tan\alpha = \frac{4}{3}$ and $\mu = 0.5$.

A horizontal force H N, in the vertical plane through a line of greatest slope, holds it in equilibrium. Find the range of possible values of H.

$\sin\alpha = \frac{4}{5}$, $\cos\alpha = \frac{3}{5}$. Normal reaction: $R = 125\cos\alpha + H\sin\alpha$

About to slide down: $H\cos\alpha + \mu R = 125\sin\alpha$, so $H(\frac{3}{5} + 0.5 \times \frac{4}{5}) = 125(\frac{4}{5} - 0.5 \times \frac{3}{5})$

$H_{min} = 62.5$

About to slide up: $H\cos\alpha = 125\sin\alpha + \mu R$, so $H(\frac{3}{5} - 0.5 \times \frac{4}{5}) = 125(\frac{4}{5} + 0.5 \times \frac{3}{5})$

$H_{max} = 687.5$

Answer: $62.5 \leq H \leq 687.5$

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10 Separable differential equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$\frac{dy}{dx} = 6xy$, and y = 3 when x = 0.

Solve the differential equation and find y when x = 2.

$\int \frac{1}{y}\, dy = \int 6x\, dx$, so $\ln y = 6x^2/2 + c = 3x^2 + c$

x = 0, y = 3: c = ln 3, so $y = 3\, e^{3x^2}$

x = 2: exponent = 12

Answer: $y = 3\, e^{3x^2}$; $y = 3e^{12}$