Day 336 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Combined mean and standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Class Q has 24 students with mean mark 34 and standard deviation 3. Class P has 25 students with mean mark 72 and standard deviation 8.
(a) Find the mean mark of all 49 students.
(b) Find the standard deviation of all 49 students.

Σx for Q = $24 \times 34 = 816$, Σx for P = $25 \times 72 = 1800$

Combined mean = $\frac{2616}{49} = 53.4$

$\Sigma x^2 = n(\sigma^2 + \text{mean}^2)$: $24(9 + 1156) + 25(64 + 5184) = 159160$

Variance = $\frac{159160}{49} - 53.388^2 = 397.91$, so $\sigma = 19.95$

Answer: (a) 53.4 (b) 19.9

2 Cubic graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the cubic y = f(x), which touches the x-axis once. Find f(x), giving your answer in expanded form.

Crosses at $x = -4$ and touches at $x = -3$, so $f(x) = k(x + 4)(x + 3)^2$.
At (0, 36): $36 = k(4)(3)^2 = 36k$, so $k = 1$.
$f(x) = (x + 4)(x + 3)^2$, then expand.
Answer: $f(x) = x^3 + 10x^2 + 33x + 36$

3 Stationary points of x^n times a root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = 3x^2\sqrt{80 - x}$, for $0 < x < 80$, has one stationary point. Find its exact coordinates and determine its nature.

$\frac{dy}{dx} = \frac{3x(320 - 5x)}{2\sqrt{80 - x}}$
$\frac{dy}{dx} = 0$: $320 - 5x = 0$, so $x = 64$
$y = 3 \times 64^2 \times 4$
Gradient changes from positive to negative: a maximum
Answer: (64, 49152), maximum

4 Lift, passenger and cable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A lift of mass M kg, carrying a passenger of mass 65 kg, hangs from a vertical cable. The tension in the cable is 4429 N and the floor exerts a force of 559 N on the passenger.
Find (a) the acceleration of the lift, with its direction, (b) the value of M. Take g = 9.8 m s^{-2}.

Passenger (upwards positive): $559 - 65 \times 9.8 = 65a$, so $a = -1.2$
$a = 1.2$ m s^{-2}, so the acceleration is downwards
Lift and passenger: $4429 - (M + 65) \times 9.8 = (M + 65) \times -1.2$
$(M + 65)(9.8 - 1.2) = 4429$, so $M + 65 = \frac{4429}{8.6} = 515$
Answer: (a) 1.2 m s^{-2} downwards (b) M = 450

5 Cubic with only one real root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 - 4x^2 + 7x - 12$.
Show that $(x - 3)$ is a factor of f(x), and that the equation $f(x) = 0$ has exactly one real root.

$f(3) = 27 - 36 + 21 - 12 = 0$, so $(x - 3)$ is a factor.
$f(x) = (x - 3)(x^2 - x + 4)$.
For $x^2 - x + 4 = 0$: $b^2 - 4ac = 1 - 16 = -15 < 0$.
No real roots from the quadratic, so the only real root is $x = 3$.
Answer: Proved; the only real root is x = 3

6 Beam on two supports: where can a person stand (AQA, Edexcel, OCR, MEI)

A uniform plank AB of length 6 m and mass 60 kg rests on supports at C and D, where AC = 2 m and DB = 2 m. A girl of mass 40 kg stands on it at distance x m from A.

Find the range of x for which the plank stays in equilibrium on both supports.

Equilibrium on both supports needs $R_C \geq 0$ and $R_D \geq 0$. Limiting cases: $R_C = 0$ and $R_D = 0$ in turn

Tilting about D (girl beyond D): $60(1) = 40(x - 4)$, so $x = 4 + \frac{60 \times 1}{40} = 5.5$

Tilting about C (girl before C): $60(1) = 40(2 - x)$, so $x = 2 - \frac{60 \times 1}{40} = 0.5$

Answer: $0.5 \leq x \leq 5.5$

7 Implicit differentiation: tangent or normal (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^3 + xy^2 = 75$.

Find the equation of the tangent to C at the point (3, 4),
in the form $ax + by + c = 0$ with integers a, b, c.

Differentiate implicitly with respect to x

At (3, 4): $dy/dx = -\frac{43}{24}$

$y - 4 = (-\frac{43}{24})(x - 3)$, then clear fractions

Answer: $43x + 24y - 225 = 0$

8 Rational inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\frac{3x - 1}{-x - 1} < -2$.

Rearrange: $\frac{3x - 1}{-x - 1} - (-2) < 0$.

Combine: $\frac{x - 3}{-x - 1} < 0$.

Critical values: $x = 3$ (zero of numerator), $x = -1$ (excluded, denominator zero).
Sign analysis of the three regions gives the solution.

Answer: $x < -1$ or $x > 3$

9 Tree diagram with an unknown probability (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The tree diagram shows two games played in turn.

Given the second is a W, P(first was a W) = $\frac{4}{7}$. Find x.

P(W then W) = 0.8x, P(second W) = $0.8x + 0.2(1 - x)$

P(first W | second W) = $\frac{0.8x}{0.8x + 0.2(1 - x)} = \frac{4}{7}$

$0.8x = \frac{4}{7}(0.8x + 0.2 - 0.2x)$

Solve the linear equation for x: $x = \frac{1}{4}$

Answer: $x = \frac{1}{4}$

10 Separable differential equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A population N satisfies $\frac{dN}{dt} = -0.005N^2$, with $N = 50$ when $t = 0$.

Find N when $t = 16$, and the time taken for N to halve.

$\int N^{-2}\, dN = -0.005 \int dt$ gives $-\frac{1}{N} = -0.005t + c$

$t = 0$: $c = -\frac{1}{50}$, so $\frac{1}{N} = 0.005t + \frac{1}{50}$

$t = 16$: $\frac{1}{N} = \frac{2}{25} + \frac{1}{50}$; halving: $0.005t = \frac{1}{50}$

Answer: $N = 10$; $t = 4$