Day 334 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Circle touching both axes (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A circle has its centre in the first quadrant and touches both the x-axis and the y-axis. It passes through the point (6, 12). Find the two possible equations of the circle.

Centre (r, r), radius r: $(x - r)^2 + (y - r)^2 = r^2$.

Substitute (6, 12): $(6 - r)^2 + (12 - r)^2 = r^2$.

$r^2 - 36r + 180 = 0$.

$(r - 6)(r - 30) = 0$, so r = 6 or r = 30.

Answer: $(x - 6)^2 + (y - 6)^2 = 36$ or $(x - 30)^2 + (y - 30)^2 = 900$

2 Velocity-time graph with a return (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the velocity-time graph of a particle P moving on a straight line (velocity in m s^{-1}, time in s). At time 15 s, P is back at its starting point.

Find (a) the value of W, (b) the greatest distance of P from its starting point.

Net area = 0 (back at start): $\frac{1}{2} \times 2 \times 12 + \frac{1}{2}(12 - W)(5) - W(8) = 0$

$12 + 30 - 2.5W - 8W = 0$, so $10.5W = 42$

W = 4

v = 0 where the line from (2, 12) to (7, −4) crosses the axis: $t = 2 + 5 \times \frac{12}{16} = 5.75$

Greatest distance = area of triangle = $\frac{1}{2} \times 5.75 \times 12$

Answer: (a) W = 4 (b) 34.5 m

3 Outliers and the 1.5 × IQR rule (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A data set has lower quartile 21 and upper quartile 36. An outlier is a value more than 1.5 × IQR beyond the nearest quartile.

(a) Find the range of values that are not outliers.

(b) 15 observations have mean 27. The largest is 63, an outlier, and it is removed. Find the new mean.

IQR = 36 − 21 = 15, so 1.5 × IQR = 22.5

Lower limit = 21 − 22.5 = −1.5, upper limit = 36 + 22.5 = 58.5

$\Sigma x = 15 \times 27 = 405$, so without 63: $\Sigma x = 342$

New mean = $\frac{342}{14}$

Answer: (a) −1.5 to 58.5 (b) 24.43

4 Newton cooling model (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A drink cools according to $\theta = 10 + Ae^{-kt}$, where θ °C is the temperature after t minutes. Initially θ = 100, and after 10 minutes θ = 55.

Find A and k, then find when θ = 36 (3 s.f.).

t = 0: 10 + A = 100 so A = 90

t = 10: $10 + 90e^{-10k} = 55$ so $e^{-10k} = \frac{1}{2}$

$k = \frac{1}{10} \ln 2 = 0.0693$

$90\, e^{-kt} = 26$ so $t = \frac{1}{k} \ln \frac{45}{13}$

t = 17.9 minutes

Answer: A = 90, k = 0.0693, t = 17.9 minutes

5 Car, trailer and tow bar (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A car of mass 1500 kg tows a trailer of mass 500 kg along a horizontal road, with driving force 2800 N. Resistances are 700 N on the car and 500 N on the trailer.

(a) Find the tension in the tow bar. At 6 m s^{-1} the tow bar breaks. (b) Find the distance between car and trailer when the trailer stops.

Whole system: $a = \frac{2800 - 700 - 500}{2000} = 0.8$

Trailer: $T - 500 = 500 \times 0.8$, so $T = 900$

After the break the trailer decelerates at $\frac{500}{500} = 1$, and stops after $t = \frac{6}{1} = 6$ s

Trailer travels $\frac{6}{2} \times 6 = 18$ m

Car: $a = \frac{2800 - 700}{1500} = 1.4$, so $s = 6 \times 6 + ½ \times 1.4 \times 6^2 = 61.2$ m

Distance apart = $61.2 - 18$

Answer: (a) 900 N (b) 43.2 m

6 Implicit differentiation: tangent or normal (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^3 + xy^2 = 40$.

Find the equation of the normal to C at the point (2, 4),

in the form $ax + by + c = 0$ with integers a, b, c.

Differentiate implicitly with respect to x

At (2, 4): $dy/dx = -\frac{7}{4}$

Normal gradient = $-1 \div (-\frac{7}{4}) = \frac{4}{7}$

$y - 4 = \frac{4}{7}(x - 2)$, then clear fractions

Answer: $4x - 7y + 20 = 0$

7 Proof by contradiction (irrationality) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Given that $\sqrt{6}$ is irrational, prove by contradiction that $\sqrt{3} + \sqrt{2}$ is irrational.

Assume $\sqrt{3} + \sqrt{2} = r$, where r is rational.

Square both sides: $3 + 2 + 2\sqrt{6} = r^2$.

So $\sqrt{6} = \frac{r^2 - 5}{2}$.

The right-hand side is rational, but $\sqrt{6}$ is irrational.

This is a contradiction, so $\sqrt{3} + \sqrt{2}$ is irrational.

Answer: Proved

8 Perpendicular from a point to a line (3D) (CIE (9709); Further Maths on other boards)

The line l has equation $\mathbf{r} = \begin{pmatrix} 6 \\ 3 \\ -4 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

The point P has coordinates (7, 5, −2).

Find the coordinates of the foot F of the perpendicular from P to l, and the exact distance PF.

F = (a + td) for some t, so $\mathrm{FP} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - t\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

FP · d = 0: $3 - t(3) = 0$, so $t = 1$

F = (5, 4, −3)

$\mathrm{PF} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, so $\mathrm{PF} = \sqrt{6}$

Answer: F = (5, 4, −3), PF = $\sqrt{6}$

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9 Reciprocal graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the curve C, its asymptotes (dashed) and a point P. C has equation $y = \frac{a}{x + p} + q$. Find a, p and q.

Asymptotes $x = -3$ and $y = 3$, so $p = 3$ and $q = 3$.

$y = \frac{a}{x + 3} + 3$.

P$(-1, 5)$: $5 = \frac{a}{2} + 3$, so $a = 4$.

Answer: a = 4, p = 3, q = 3

10 Continuous random variable: pdf with unknown constant (CIE)

The pdf of X is $f(x) = \frac{k}{x^2}$ for $1 \leq x \leq 8$, and 0 otherwise.

(a) Find k.
(b) Find the median of X.
(c) Find E(X), to 3 significant figures.

(a) $\int k\, x^{-2}\, dx$ from 1 to 8 $= k(1 - \frac{1}{8}) = 1$, so $k = \frac{8}{7}$

(b) $\int f(x)\, dx$ from 1 to m $= \frac{8}{7}(1 - \frac{1}{m}) = \frac{1}{2}$

$1 - \frac{1}{m} = \frac{7}{16}$, so $m = \frac{16}{9}$

(c) $E(X) = \int \frac{8}{7}\, x^{-1}\, dx = \frac{8}{7} \ln 8 = 2.38$

Answer: (a) k = $\frac{8}{7}$ (b) $\frac{16}{9}$ (c) 2.38