Day 333 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Trig equations with compound angles (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\cot(3\theta - 20°) = \frac{1}{\sqrt{3}}$
for $-180° \le \theta \le 180°$.

Let $u = 3\theta - 20°$; then $\cot u = \frac{1}{\sqrt{3}}$
Base solutions: $u = 60°$ (+ multiples of 180°)
$\theta = (u + 20°) \div 3$, keeping those in range
Answer: $\theta = -153.3°, -93.3°, -33.3°, 26.7°, 86.7°, 146.7°$

2 Beam on two supports: where can a person stand (AQA, Edexcel, OCR, MEI)

A uniform plank AB of length 8 m and mass 30 kg rests on supports at C and D, where AC = 3 m and DB = 2 m. A girl of mass 60 kg stands on it at distance x m from A.
Find the range of x for which the plank stays in equilibrium on both supports.

Equilibrium on both supports needs $R_C \ge 0$ and $R_D \ge 0$. Limiting cases: $R_C = 0$ and $R_D = 0$ in turn

Tilting about D (girl beyond D): $30(2) = 60(x - 6)$, so $x = 6 + \frac{30 \times 2}{60} = 7$

Tilting about C (girl before C): $30(1) = 60(3 - x)$, so $x = 3 - \frac{30 \times 1}{60} = 2.5$

Answer: $2.5 \le x \le 7$

3 Geometric series: least n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A geometric series has first term 4 and common ratio $\frac{4}{5}$.
Find the least n for which S_n exceeds 99% of the sum to infinity.

$S_n = S_\infty(1 - r^n) > 0.99\, S_\infty$.
$1 - \frac{4}{5}^n > 0.99$, so $\frac{4}{5}^n < 0.01$.
$n > \frac{\log(0.01)}{\log(0.8)} = 20.638$.
So the least n is 21.
Answer: n = 21

4 Hypothesis test for a normal mean (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The fill volume of a bottle is $N(\mu, 25)$ with known variance 25. The mean was claimed to be 30 ml.
A random sample of 36 has mean 30.5. Test at the 1% level whether the mean is less than 30.

$H_0: \mu = 30$, $H_1: \mu < 30$. Under H_0, the sample mean ~ $N(30, \frac{25}{36})$
$z = \frac{30.5 - 30}{5 \div \sqrt{36}} = 0.600$
p-value = $P(Z < 0.600) = 0.726$
$0.726 > 1\%$, so do not reject H_0
There is insufficient evidence at the 1% level that the mean has changed as claimed
Answer: Do not reject H_0 (z = 0.600, p-value 0.726)

5 Sigma notation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Evaluate $\sum_{r=6}^{11}(5 \times 2^r + 3r)$.

Geometric part: first term $5 \times 2^6 = 320$, ratio 2, 6 terms.
Sum = $\frac{320(2^6 - 1)}{1} = 20160$.
Arithmetic part: 3 × 6 terms, average (6 + 11)/2: sum = 153.
Total = 20160 + 153.
Answer: 20313

6 Newton cooling model (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A drink cools according to $\theta = 15 + Ae^{-kt}$, where θ °C is the temperature after t minutes. Initially $\theta = 85$, and after 10 minutes $\theta = 57$.
Find A and k, then find when $\theta = 34$ (3 s.f.).

$t = 0$: $15 + A = 85$ so $A = 70$

$t = 10$: $15 + 70e^{-10k} = 57$ so $e^{-10k} = \frac{3}{5}$

$k = \frac{1}{10} \ln \frac{5}{3} = 0.0511$

$70\, e^{-kt} = 19$ so $t = \frac{1}{k} \ln \frac{70}{19}$

$t = 25.5$ minutes

Answer: A = 70, k = 0.0511, t = 25.5 minutes

7 Car on a hill: power and maximum speed (CIE)

A car of mass 1500 kg, with engine power 35.4 kW, climbs a straight road with $\sin \alpha = \frac{1}{10}$ against a constant resistance of 300 N.
Find (a) the greatest steady speed up the hill, (b) the acceleration at 5 m s^{-1}. Take g = 9.8 m s^{-2}.

Steady speed: driving force $= 300 + 1500 \times 9.8 \times \frac{1}{10} = 1770$ N

$P = F v$, so $v = \frac{35400}{1770} = 20$

At 5 m s^{-1}: $F = \frac{35400}{5} = 7080$ N

$F - 1770 = 1500\, a$, so $a = \frac{5310}{1500}$

Answer: (a) 20 m s^{-1} (b) 3.54 m s^{-2}

8 Parametric differentiation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = 3 \sin t$, $y = 2 \cos 2t$.
Find a cartesian equation of the curve in the form $y = f(x)$.

$\cos 2t = 1 - 2\sin^2 t$

$\sin t = \frac{x}{3}$, so $y = 2\left(1 - 2 \times \frac{x^2}{9}\right)$

Answer: $y = 2 - \frac{4}{9}x^2$

9 Rational inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\frac{3}{(x+2)} > \frac{5}{(x-3)}$.

Multiply by the positive quantity $(x + 2)^2(x - 3)^2$:
$3(x - (-2))(x - 3)^2 > 5(x - 3)(x - (-2))^2$.
Move to one side and factorise: $(x - (-2))(x - 3)(-2x - 19) > 0$.

Critical values: -2, 3, $-\frac{19}{2}$. Test the four regions.

Answer: $x < -\frac{19}{2}$ or $-2 < x < 3$

10 Normal test: critical region and Type II error (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim N(\mu, 25)$. H_0: $\mu = 84$ is tested against H_1: $\mu > 84$ using the mean of a sample of size 9 at the 10% level.

(a) Find the critical region for the sample mean.

(b) Find the probability of a Type II error when $\mu = 85.5$.

Under H_0, sample mean ~ $N(84, \frac{25}{9})$, standard error = 1.667

$z = 1.2816$ for the upper tail: critical value = $84 + 1.2816 \times 1.667 = 86.14$

Type II error: do not reject H_0, sample mean < 86.14, where mean ~ N(85.5, 1.667)

$z = \frac{86.14 - 85.5}{1.667} = 0.382$

Probability = 0.649

Answer: (a) sample mean > 86.1 (b) 0.649