Day 332 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Poisson: sum of two independent variables, conditional (CIE)

X and Y are independent, X ~ Po(4) and Y ~ Po(3).
(a) State the distribution of X + Y.
(b) Given that X + Y = 6, find the probability that X = 5.

X + Y ~ Po(7)

$P(X = 5 \mid X + Y = 6) = \frac{P(X = 5)P(Y = 1)}{P(X + Y = 6)}$

$= \frac{e^{-4}4^5}{5! \times e^{-3}3^1/1!} \div \frac{e^{-7}7^6}{6!}$

$= C(6, 5)\ (\frac{4}{7})^5\ (\frac{3}{7})^1$, a binomial probability

= 0.157

Answer: (a) Po(7) (b) 0.157

2 Periodic recurrence relations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sequence is defined by $u_1 = -1$ and

$u_{n+1} = \frac{1}{1 - u_n}$.

Show that the sequence is periodic and find

$\sum_{r=1}^{75} u_r$.

$u_1 = -1$, $u_2 = \frac{1}{2}$, $u_3 = 2$, $u_4 = -1$.

$u_4 = u_1$, so the sequence repeats with period 3.

One period sums to $\frac{3}{2}$.

75 = 3 × 25 + 0: 25 full periods plus the first 0 terms.

Answer: $\frac{75}{2}$

3 Logarithmic equations with ln (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve 2 ln x = ln(x + 42) + ln 1,
stating clearly why one root is rejected.

$\ln x^2 = \ln(1(x + 42))$, so $x^2 = 1x + 42$

$x^2 - 1x - 42 = 0$, $(x - 7)(x + 6) = 0$

$x = -6$ is rejected since ln x is undefined for x < 0

Answer: x = 7

4 Velocity parallel to a given direction (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves in a plane. At time t seconds its position vector relative to a fixed origin O is r metres, where $\mathbf{r} = (-t^2)\mathbf{i} + (2.5t^2 - 6t)\mathbf{j}$.
Find (a) the value of t at which P is moving parallel to $(\mathbf{i} - 2\mathbf{j})$, (b) the speed of P at this time (3 s.f. if not exact).

$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-2t)\mathbf{i} + (5t - 6)\mathbf{j}$

Parallel to $(\mathbf{i} - 2\mathbf{j})$: $(-2t) \times -2 = (5t - 6) \times 1$

$t(-1) = -6$, so t = 6

At t = 6: $\mathbf{v} = (-12\mathbf{i} + 24\mathbf{j})$, speed $= \sqrt{(-12^2 + 24^2)}$

Answer: (a) t = 6 (b) 26.8 m s^{-1}

5 Fixed point iteration and rearrangement (AQA, Edexcel, OCR, MEI, Eduqas, CIE (iterative methods))

The equation $x^3 = x + 15$ has a root α.
(a) Show that the equation can be written as $x = \sqrt[3]{(x + 15)}$.
(b) Use $x_{n+1} = \sqrt[3]{(x_n + 15)}$ with $x_0 = 2$ to find x_1, x_2, x_3
to 4 d.p., and hence state α to 3 d.p.

Take the cube root of both sides to obtain $x = \sqrt[3]{(x + 15)}$
$x_1 = 2.5713$, $x_2 = 2.5998$, $x_3 = 2.6012$
The iteration converges (successive values agree to 3 d.p.): $\alpha = 2.601$
Answer: $x_1 = 2.5713$, $x_2 = 2.5998$, $x_3 = 2.6012$; $\alpha = 2.601$

6 Arithmetic series modelling (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A pile of logs has 10 logs in its bottom row. Each row above has 1 fewer logs than the row below, and the pile has as many rows as possible, each with at least one log. Find the number of rows and the total number of logs.

Row n has $10 - 1(n - 1) = -n + 11$ logs; need this ≥ 1.
$1(n - 1) \leq 9$, so $n \leq 10$: 10 rows.

Top row = 1 logs. $S = \frac{10}{2}(10 + 1) = 55$.

Answer: 10 rows, 55 logs

7 Tangent and axes triangle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The tangent to $y = \frac{x^2 + 5}{x + 2}$ at the point where $x = 4$
meets the axes at A and B. Find the exact area of triangle OAB,
where O is the origin.

At $x = 4$: $y = \frac{7}{2}$. Differentiate: gradient = $\frac{3}{4}$

Tangent: $y - \frac{7}{2} = \frac{3}{4}(x - 4)$, so $y = \frac{3}{4}x + \frac{1}{2}$

y-intercept $\frac{1}{2}$; x-intercept $-\frac{2}{3}$

Area = $\frac{1}{2} \times |\frac{1}{2}| \times |-\frac{2}{3}|$

Answer: $\frac{1}{6}$

8 Relative motion of two projectiles (AQA, Edexcel, OCR, MEI, Eduqas)

Particles A and B are projected from a point O at the same instant, with initial velocities $(16\mathbf{i} + 40\mathbf{j})$ m s^{-1} and $(8\mathbf{i} + 34\mathbf{j})$ m s^{-1} respectively (i horizontal, j vertically upwards).
Both are still in flight after 2 s. Find (a) the distance AB at this time, (b) the angle AB makes with the horizontal, to 1 d.p.

Position of A: $16t\mathbf{i} + (40t - 4.9t^2)\mathbf{j}$; position of B: $8t\mathbf{i} + (34t - 4.9t^2)\mathbf{j}$
Subtracting, the $4.9t^2$ terms cancel: $AB = t(8\mathbf{i} + 6\mathbf{j})$
After 2 s: $AB = (16\mathbf{i} + 12\mathbf{j})$
Distance = $\sqrt{(16^2 + 12^2)} = 20$

The direction of AB is that of $(8\mathbf{i} + 6\mathbf{j})$, which does not depend on t: tan angle = $\frac{6}{8}$

Answer: (a) 20 m (b) 36.9°

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9 Most likely value of a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(10, \frac{1}{5})$.

(a) Show that $P(X = r + 1) \div P(X = r) = \frac{10 - r}{r + 1} \times \frac{1}{4}$.

(b) Find the most likely value of X and its probability (3 s.f.).

$P(X = r + 1) \div P(X = r) = \frac{C(10, r + 1)}{C(10, r)} \times \frac{p}{q} = \frac{10 - r}{r + 1} \times \frac{p}{q}$

With $p = \frac{1}{5}$ and $q = \frac{4}{5}$, $p \div q = \frac{1}{4}$

The ratio exceeds 1 when $(10 - r)\frac{1}{5} > (r + 1)\frac{4}{5}$, that is $r < 1.2$

So probabilities rise up to r = 2 and then fall; the mode is 2

P(X = 2) = 0.302

Answer: 2 with probability 0.302

10 Perpendicular lines and area (diagram) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

In the diagram, line l_1 passes through A and B.

Line l_2 is perpendicular to l_1 at B and meets the

x-axis at C. Find the coordinates of C and the area of triangle ABC.

Gradient of $l_1 = \frac{9 - 3}{-3 - 6} = -\frac{2}{3}$.

Gradient of $l_2 = \frac{3}{2}$. Equation: $y - 9 = \frac{3}{2}(x + 3)$.

y = 0: x = −9, so C(−9, 0).

AB = $3\sqrt{13}$, BC = $3\sqrt{13}$: area = ½ × AB × BC.

Answer: C(−9, 0); area $\frac{117}{2}$