Day 330 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Double angle equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\cos 2\theta - 3\sin\theta - 2 = 0$
for $0 \leq \theta < 360°$.

Use $\cos 2\theta \equiv 1 - 2\sin^2\theta$: $2\sin^2\theta + 3\sin\theta + 1 = 0$
$(2\sin\theta + 1)(\sin\theta + 1) = 0$

$\sin\theta = -\frac{1}{2}$ or $\sin\theta = -1$

Using the CAST diagram for $0 \leq \theta < 360°$
Answer: θ = 210°, 270°, 330°

2 Normal: unknown standard deviation then a second probability (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The scores in an aptitude test are modelled by $N(48, \sigma^2)$. The probability of a value above 66 is 0.05.
(a) Find σ.
(b) Find the probability of a value above 67.

$P(X > 66) = 0.05$ so $Z = \frac{66 - 48}{\sigma} = 1.6449$

$\sigma = \frac{18}{1.6449} = 10.94$

$z = \frac{67 - 48}{10.94} = 1.736$

$P(X > 67) = P(Z > 1.736) = 0.0413$
Answer: (a) 10.9 (b) 0.0413

3 Surds: nested radicals (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\sqrt{64 + 24\sqrt{7}}$ in the form
$a + b\sqrt{7}$, where a and b are positive integers.

Suppose $\sqrt{64 + 24\sqrt{7}} = a + b\sqrt{7}$. Square both sides.
$64 + 24\sqrt{7} = a^2 + 7b^2 + 2ab\sqrt{7}$.
Equate: $a^2 + 7b^2 = 64$ and $2ab = 24$, so $ab = 12$.
Try $a = 6$, $b = 2$: $36 + 28 = 64$ ☐.
($a > b\sqrt{7}$, so the root is positive.)
Answer: $6 + 2\sqrt{7}$

4 Binomial: mean and variance given (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A binomial random variable X has mean 6 and variance 5.4.
(a) Find n and p.
(b) Find P(X = 4), correct to 3 significant figures.

$np = 6$ and $np(1 - p) = 5.4$

Divide: $1 - p = 5.4 \div 6 = \frac{9}{10}$, so $p = \frac{1}{10}$

$n = 6 \div \frac{1}{10} = 60$

$P(X = 4) = C(60, 4)\left(\frac{1}{10}\right)^4\left(\frac{9}{10}\right)^{56} = 0.134$

Answer: (a) n = 60, p = $\frac{1}{10}$ (b) 0.134

5 Range and greatest height to speed and angle (AQA, Edexcel, OCR, MEI, Eduqas)

A golf ball is hit from a point O on horizontal ground. It lands 160 m from O, and reaches a maximum height of 30 m above the ground.

Find the angle of projection and the speed of projection, to 3 s.f. Take g = 9.8 m s^{-2}.

Vertical: $0 = u_y^2 - 2 \times 9.8 \times 30$, so $u_y = 24.25$

Time of flight = $\frac{2u_y}{9.8} = 4.949$ s, so $u_x = \frac{160}{4.949} = 32.33$

$\tan\theta = \frac{u_y}{u_x} = 0.7500$, so $\theta = 36.9°$

Speed = $\sqrt{(u_x^2 + u_y^2)} = 40.4$

Answer: θ = 36.9°, speed 40.4 m s^{-1}

6 Tangent and axes triangle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The tangent to $y = x\sqrt{4x - 4}$ at the point where x = 2 meets the axes at A and B. Find the exact area of triangle OAB, where O is the origin.

At x = 2: y = 4. Differentiate: gradient = 4

Tangent: $y - 4 = 4(x - 2)$, so $y = 4x - 4$

y-intercept −4; x-intercept 1

Area = $\frac{1}{2} \times |-4| \times |1|$

Answer: 2

7 Quadratic sequence from recurrence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = u_n + n + 2$, $u_1 = 3$.

(a) Show that $u_n = an^2 + bn + c$ and find a, b, c.

(b) Find the least n for which $u_n > 1346$.

$u_n = u_1 + \sum_{k=1}^{n-1}(k + 2) = 3 + 1 \times \frac{(n - 1)n}{2} + 2(n - 1)$.

$= \frac{1}{2}n^2 + \frac{3}{2}n + 1$.

So $a = \frac{1}{2}$, $b = \frac{3}{2}$, $c = 1$.

$u_{50} = 1326 \leq 1346$ and $u_{51} = 1378 > 1346$.

Answer: (a) a = $\frac{1}{2}$, b = $\frac{3}{2}$, c = 1 (b) n = 51

8 Cubic graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the cubic y = f(x), which touches the x-axis once. Find f(x), giving your answer in expanded form.

Crosses at x = −1 and touches at x = 2, so $f(x) = k(x + 1)(x - 2)^2$.

At (0, 4): $4 = k(1)(-2)^2 = 4k$, so k = 1.

$f(x) = (x + 1)(x - 2)^2$, then expand.

Answer: f(x) = $x^3 - 3x^2 + 4$

9 Exponential times trigonometric (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$y = e^{-3x} \sin 2x$. Show that $dy/dx = e^{-3x}(-3 \sin 2x + 2 \cos 2x)$.

Find the x coordinate of the stationary point with the smallest positive x (3 s.f.), and its nature.

Product rule: $dy/dx = (-3)e^{-3x} \sin 2x + e^{-3x} \times 2 \cos 2x$

$dy/dx = 0$ requires $-3 \sin 2x + 2 \cos 2x = 0$

$\tan 2x = \frac{2}{3}$, smallest positive 2x = 0.5880

x = 0.294

Second derivative test (or gradient change): a maximum

Answer: x = 0.294, maximum

10 Particle on a smooth slope joined to hanging particle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particles A (1 kg) and B (5 kg) are joined by a light string over a smooth pulley (see diagram). A lies on a smooth slope, $\alpha = 30°$, and B is 3 m above the ground.

Released from rest, find the further distance A moves up the slope after B lands (A does not reach the pulley).

A: $T - 1 \times 9.8 \times \frac{1}{2} = 1\,a$. B: $5 \times 9.8 - T = 5\,a$

Adding: $9.8(5 - 1 \times \frac{1}{2}) = 6\,a$, so $a = 7.35$

$T = 5(9.8 - 7.35) = 12.25$

When B lands: $v^2 = 2 \times 7.35 \times 3 = 44.1$

Then A decelerates at $g \sin \alpha = 4.9$: further distance $= \frac{44.1}{2 \times 4.9} = 4.5$

Answer: 4.5 m