Day 33 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Partial fractions (repeated factor) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{-5x^2 + 32x - 56}{(x-4)(x-2)^2}$ in the form $\frac{A}{(x-4)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$.

$-5x^2 + 32x - 56 \equiv A(x-2)^2 + B(x-4)(x-2) + C(x-4)$.

Put $x = 4$: $-8 = A(2)^2$, so $A = -2$.

Put $x = 2$: $-12 = C(-2)$, so $C = 6$.

Compare x^2: $-5 = 1A + 1B$, so $B = -3$.

Answer: A = −2, B = −3, C = 6

2 Car, trailer and tow bar (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A car of mass 1200 kg tows a trailer of mass 400 kg along a horizontal road, with driving force 1120 N. Resistances are 400 N on the car and 240 N on the trailer.

(a) Find the tension in the tow bar. At 4.8 m s^{-1} the tow bar breaks. (b) Find the distance between car and trailer when the trailer stops.

Whole system: $a = \frac{1120 - 400 - 240}{1600} = 0.3$

Trailer: $T - 240 = 400 \times 0.3$, so $T = 360$

After the break the trailer decelerates at $\frac{240}{400} = 0.6$, and stops after $t = \frac{4.8}{0.6} = 8$ s

Trailer travels $\frac{4.8}{2} \times 8 = 19.2$ m

Car: $a = \frac{1120 - 400}{1200} = 0.6$, so $s = 4.8 \times 8 + \frac{1}{2} \times 0.6 \times 8^2 = 57.6$ m

Distance apart = $57.6 - 19.2$

Answer: (a) 360 N (b) 38.4 m

3 Outliers and the 1.5 × IQR rule (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A data set has lower quartile 16 and upper quartile 23. An outlier is a value more than 1.5 × IQR beyond the nearest quartile.

(a) Find the range of values that are not outliers.

(b) 8 observations have mean 17. The largest is 38, an outlier, and it is removed. Find the new mean.

IQR = $23 - 16 = 7$, so $1.5 \times \text{IQR} = 10.5$

Lower limit = $16 - 10.5 = 5.5$, upper limit = $23 + 10.5 = 33.5$

$\Sigma x = 8 \times 17 = 136$, so without 38: $\Sigma x = 98$

New mean = $\frac{98}{7}$

Answer: (a) 5.5 to 33.5 (b) 14.00

4 Newton-Raphson iteration (AQA, Edexcel, OCR, MEI, Eduqas (Newton-Raphson))

$f(x) = x^3 + x - 15$.

(a) Show that $f(x) = 0$ has a root α with $2 < \alpha < 3$.

(b) Use Newton-Raphson with $x_0 = 2$ to find x_1 and x_2 to 4 decimal places.

$f(2) = -5$ and $f(3) = 15$: a change of sign, f continuous

$f'(x) = 3x^2 + 1$, so $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x_1 = 2 - \frac{-5.000}{13.00} = 2.3846$

$x_2 = 2.3846 - \frac{0.9445}{18.06} = 2.3323$

Answer: $x_1 = 2.3846$, $x_2 = 2.3323$

5 Discrete random variable with a probability formula (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$P(X = x) = kx^2$ for x = 1, 2, ..., 4, and 0 otherwise.

(a) Show that $k = \frac{1}{30}$.

(b) Find E(X) and Var(X).

(c) Find Var(4X + 4).

(a) $\Sigma P(X = x) = k(1 + 4 + 9 + 16) = 30k = 1$, so $k = \frac{1}{30}$

(b) $E(X) = \Sigma x P(X = x) = \frac{1\times1 + 2\times4 + 3\times9 + 4\times16}{30} = \frac{10}{3}$

$E(X^2) = \frac{1\times1 + 4\times4 + 9\times9 + 16\times16}{30} = \frac{59}{5}$

$Var(X) = \frac{59}{5} - \left(\frac{10}{3}\right)^2 = \frac{31}{45}$

(c) $Var(4X + 4) = 4^2 Var(X) = 16 \times \frac{31}{45} = \frac{496}{45}$

Answer: (b) $E(X) = \frac{10}{3}$, $Var(X) = \frac{31}{45}$ (c) $\frac{496}{45}$

6 Stationary points: exact coordinates (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact coordinates of the stationary point of

$y = \frac{e^x}{x^3}$ for x > 0,

and determine its nature.

$dy/dx = \frac{e^x(x - 3)}{x^4} = 0$ gives x = 3

$y = \frac{1}{27}e^3$ when x = 3

Second derivative at this point is positive, so it is a minimum

Answer: $(3, \frac{1}{27}e^3)$, minimum

7 Reciprocal graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the curve C, its asymptotes (dashed) and a point P. C has equation

$y = \frac{a}{x + p} + q$. Find a, p and q.

Asymptotes x = 3 and y = 2, so p = −3 and q = 2.

$y = \frac{a}{x - 3} + 2$.

P(2, 8): $8 = \frac{a}{-1} + 2$, so a = −6.

Answer: a = −6, p = −3, q = 2

8 Conditions on the roots of a quadratic (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

One root of $x^2 + kx + 27 = 0$ is the square of the other root. Find the possible values of k.

Let the roots be α and α^2.

Product: $\alpha^3 = 27$, so α = 3 (the only real cube root).

Sum: $\alpha + \alpha^2 = 3 + 9 = 12 = -k$.

Answer: k = −12

9 Boat crossing a river (AQA, Edexcel, OCR, MEI)

A river is 60 m wide and flows at 15 m s^{-1}. A boatman can row at 17 m s^{-1} in still water and wants to reach the point directly opposite his starting point.
Find (a) the angle to the bank at which he should head, to 1 d.p., (b) the time taken, to 3 s.f.

Resultant velocity must be perpendicular to the bank, so the upstream component of the boat's velocity cancels the current
Let the boat head at angle φ upstream from the perpendicular: $17 \sin \varphi = 15$, so $\varphi = 61.9°$
Angle to the bank = $90° - \varphi = 28.1°$
Speed across = $\sqrt{(17^2 - 15^2)} = 8$ m s^{-1}
Time = $\frac{60}{8} = 7.50$ s

Answer: (a) 28.1° (b) 7.50 s

10 R-form: greatest and least values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(\theta) = \frac{3}{31 + 7\sin\theta + 24\cos\theta}$
(a) Write $7\sin\theta + 24\cos\theta$ as $R\sin(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.
(b) Find the greatest and least values of $f(\theta)$, and the smallest positive θ at which the greatest value occurs, to 1 d.p.

$R = \sqrt{7^2 + 24^2} = 25$, $\tan\alpha = \frac{24}{7}$, $\alpha = 73.7°$
Denominator = $31 + 25\sin(\theta + 73.7°)$, which lies between 6 and 56
Greatest $f(\theta) = \frac{3}{6} = \frac{1}{2}$, when $\sin(\theta + \alpha) = -1$
Least $f(\theta) = \frac{3}{56} = \frac{3}{56}$, when $\sin(\theta + \alpha) = 1$
$\sin(\theta + \alpha) = -1$: $\theta + 73.7° = 270°$, so $\theta = 196.3°$

Answer: (a) $25\sin(\theta + 73.7°)$ (b) greatest $\frac{1}{2}$, least $\frac{3}{56}$, $\theta = 196.3°$