Day 329 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Particle on a rough table, string slack when B lands (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particles A (9 kg) and B (3 kg) are joined by a light string over a smooth pulley (see diagram). A lies on a rough table with $\mu = 0.25$, and B is 1.6 m above the floor.
The system is released from rest. Find the total distance A moves before coming to rest (A does not reach the pulley).

Until B lands, A: $T - 0.25 \times 9 \times 9.8 = 9a$. B: $3 \times 9.8 - T = 3a$
Adding: $(3 - 0.25 \times 9) \times 9.8 = 12a$, so $a = 0.6125$
$v^2 = 2 \times 0.6125 \times 1.6 = 1.96$, so $v = 1.4$

After B lands, A decelerates at $\mu g = 2.45$: extra distance $= \frac{1.4^2}{2 \times 2.45} = 0.4$

Total distance $= 1.6 + 0.4$
Answer: 2 m

2 Most likely value of a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(23, \frac{3}{10})$.

(a) Show that $P(X = r + 1) \div P(X = r) = \frac{23 - r}{r + 1} \times \frac{3}{7}$.

(b) Find the most likely value of X and its probability (3 s.f.).

$P(X = r + 1) \div P(X = r) = \frac{C(23, r + 1)}{C(23, r)} \times \frac{p}{q} = \frac{23 - r}{r + 1} \times \frac{p}{q}$

With $p = \frac{3}{10}$ and $q = \frac{7}{10}$, $p \div q = \frac{3}{7}$

The ratio exceeds 1 when $(23 - r)\frac{3}{10} > (r + 1)\frac{7}{10}$, that is $r < 6.2$

So probabilities rise up to $r = 7$ and then fall; the mode is 7
$P(X = 7) = 0.178$
Answer: 7 with probability 0.178

3 Related rates of change (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ladder of length 15 m leans against a vertical wall. Its foot slides away from the wall at 3 m s^{-1}. When the foot is 9 m from the wall, find the rate at which the top is moving down.

$x^2 + y^2 = 225$, so $2x\,dx/dt + 2y\,dy/dt = 0$
When $x = 9$, $y = 12$
$dy/dt = -\frac{x}{y}\,dx/dt = -\frac{9}{12} \times 3$
Answer: $\frac{9}{4}$ m s^{-1}

4 Self-inverse function (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{kx - 8}{x - 6}$ for $x \neq 6$. Given that f is self-inverse, find k, and hence solve $f(x) = x$.

Let $y = f(x)$: $y(x - 6) = kx - 8$, so $x(y - k) = 6y - 8$.
$f^{-1}(x) = \frac{6x - 8}{x - k}$.
$f = f^{-1}$ requires $k = 6$.
$f(x) = x$: $6x - 8 = x(x - 6)$, so $x^2 - 12x + 8 = 0$.
$x = 6 \pm \sqrt{28} = 6 \pm 2\sqrt{7}$.
Answer: $k = 6$; $x = 6 \pm 2\sqrt{7}$

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5 Direction of motion at 45 degrees (AQA, Edexcel, OCR, MEI, Eduqas)

A stone is projected from level ground with horizontal and vertical velocity components 19.6 and 63.7 m s^{-1}.
Find (a) the two times at which it moves at 45° to the horizontal, (b) the horizontal distance between those two points.
Take g = 9.8 m s^{-2}.

At 45° to the horizontal, |v_y| = v_x = 19.6
Moving upwards: 63.7 − 9.8t = 19.6, so t = 4.5
Moving downwards: 63.7 − 9.8t = −19.6, so t = 8.5
Horizontal distance = 19.6 × (8.5 − 4.5)
Answer: (a) 4.5 s and 8.5 s (b) 78.4 m

6 Quadratic sequence from recurrence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = u_n + 3n$, $u_1 = 4$.
(a) Show that $u_n = an^2 + bn + c$ and find a, b, c.
(b) Find the least n for which $u_n > 1041$.

$$u_n = u_1 + \sum_{k=1}^{n-1}(3k) = 4 + 3 \times \frac{(n-1)n}{2} + 0(n-1).$$

$$= \frac{3}{2}n^2 - \frac{3}{2}n + 4.$$

So $a = \frac{3}{2}$, $b = -\frac{3}{2}$, $c = 4$.

$u_{26} = 979 \leq 1041$ and $u_{27} = 1057 > 1041$.

Answer: (a) $a = \frac{3}{2}$, $b = -\frac{3}{2}$, $c = 4$ (b) $n = 27$

7 Implicit curves: horizontal and vertical tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^2 - 3xy + 5y^2 = 44$.
Find the coordinates of the points on C where the tangent is parallel to the x-axis.

Differentiate: $2x + by + (bx + 2cy)dy/dx = 0$, so $dy/dx = -\frac{2x + -3y}{-3x + 10y}$

Horizontal tangent when $2x + -3y = 0$, so $y = \frac{2x}{3}$

Substitute into the equation of C: $x^2 + (-3xy) + (5y^2) = 44$
Solve for the remaining coordinate
Answer: (−6, −4) and (6, 4)

8 Parametric to Cartesian (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations

$x = 2(t + \frac{1}{t})$, $y = (t - \frac{1}{t})$, $t \neq 0$.

Find a Cartesian equation of the curve.

$$x^2 = 2^2(t^2 + 2 + \frac{1}{t^2}).$$

$$y^2 = (t^2 - 2 + \frac{1}{t^2}).$$

$$\frac{x^2}{4} - \frac{y^2}{1} = 4.$$

Answer: $\frac{x^2}{4} - \frac{y^2}{1} = 4$

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9 Hypothesis test for a normal mean (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The daily output of a machine is N(μ, 4) with known variance 4. The mean was claimed to be 82 units. A random sample of 16 has mean 83.5. Test at the 5% level whether the mean is less than 82.

H_0: $\mu = 82$, H_1: $\mu < 82$. Under H_0, the sample mean ~ $N(82, \frac{4}{16})$

$z = \frac{83.5 - 82}{2 \div \sqrt{16}} = 3.000$

p-value = $P(Z < 3.000) = 0.999$

$0.999 > 5\%$, so do not reject H_0

There is insufficient evidence at the 5% level that the mean has changed as claimed

Answer: Do not reject H_0 (z = 3.000, p-value 0.999)

10 Partial fractions then integrate (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Express $\frac{-x}{(x + 2)(x + 1)}$ in partial fractions.

(b) Hence find the exact value of $\int_0^1 \frac{-x}{(x + 2)(x + 1)}\,dx$.

Let N/D = $\frac{A}{x + 2} + \frac{B}{x + 1}$: $-x \equiv A(x + 1) + B(x + 2)$

$x = -2$ gives $A = -2$; $x = -1$ gives $B = 1$

Integral = $[-2 \ln(x + 2) + 1 \ln(x + 1)]$

Substitute limits 0 and 1

Answer: (a) $-2/(x + 2) + 1/(x + 1)$; (b) $3\ln 2 - 2\ln 3$