Day 328 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Interception at constant speed (AQA, Edexcel, OCR, MEI, Eduqas)

At time t = 0 hours, ship A is at the point with position vector $(4\mathbf{i} + 24\mathbf{j})$ km relative to a port O, and moves with constant velocity $(3\mathbf{i})$ km h^{-1}. Ship B leaves O at t = 0 with constant speed 13 km h^{-1} and intercepts A.
Find (a) the time at which B intercepts A, (b) the velocity of B as a vector.

Position of A at time t: $(4\mathbf{i} + 24\mathbf{j}) + t(3\mathbf{i})$. B at time t: t v_B (unknown velocity)
At interception B has travelled 13 t, so $|(4\mathbf{i} + 24\mathbf{j}) + t(3\mathbf{i})| = 13\ t$
Squaring and rearranging: $-160t^2 + 24t + 592 = 0$, which gives t = 2 or a negative value
t = 2 (the other root is negative since B is faster than A)
A is then at $(4\mathbf{i} + 24\mathbf{j}) + 2(3\mathbf{i}) = (10\mathbf{i} + 24\mathbf{j})$, so $v_B = \frac{1}{2}(10\mathbf{i} + 24\mathbf{j}) = (5\mathbf{i} + 12\mathbf{j})$

Answer: (a) 2 h (b) $(5\mathbf{i} + 12\mathbf{j})$ km h^{-1}

2 Logarithmic equations with ln (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $e^x + 48\ e^{-x} = 14$,
giving your answers exactly in terms of natural logarithms.

Multiply by e^x: $e^{2x} - 14e^x + 48 = 0$
$(e^x - 6)(e^x - 8) = 0$
$e^x = 6$ or $e^x = 8$
Answer: $x = \ln 6$ and $x = \ln 8$

3 Resultant of forces: unknown constant and acceleration (AQA, Edexcel, OCR, MEI, Eduqas)

Three forces $(4\mathbf{j})$ N, $(-7\mathbf{i} + k\mathbf{j})$ N and $(-5\mathbf{i} - 6\mathbf{j})$ N act on a particle of mass 3 kg. The resultant is parallel to $(-12\mathbf{i} - 5\mathbf{j})$.
Find (a) the value of k, (b) the magnitude of the acceleration of the particle, to 3 s.f.

Resultant = $(-12)\mathbf{i} + (-2 + k)\mathbf{j}$
Parallel to $(-12\mathbf{i} - 5\mathbf{j})$: $\frac{-2 + k}{-12} = \frac{-5}{-12}$, so $k = -3$
R = $(-12\mathbf{i} - 5\mathbf{j})$ N, so $|R| = \sqrt{(-12^2 + -5^2)} = 13.00$
a = |R| ÷ 3
Answer: (a) $k = -3$ (b) 4.33 m s^{-2}

4 Addition formulae with exact values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A is acute and B is obtuse. $\cos A = \frac{7}{25}$ and $\cos B = -\frac{21}{29}$.
Find the exact values of sin(A + B) and cos(A + B).

$\cos A = \frac{7}{25}$, $\sin A = \frac{24}{25}$; $\sin B = \frac{20}{29}$, $\cos B = -\frac{21}{29}$
$\sin(A + B) = \sin A \cos B + \cos A \sin B = -\frac{364}{725}$
$\cos(A + B) = \cos A \cos B - \sin A \sin B = -\frac{627}{725}$
Answer: $\sin(A + B) = -\frac{364}{725}$, $\cos(A + B) = -\frac{627}{725}$

5 Circle through three points (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A circle passes through A(−1, −12), B(11, 4)
and C(5, 6). Find its equation.

Centre is on the perpendicular bisector of each chord.
Bisector of AB: $12x + 16y = -4$.
Bisector of BC: $-6x + 2y = -38$.
Solving: centre $(5, -4)$.
$r^2 = (-1 - 5)^2 + (-12 - (-4))^2 = 100$.
Answer: $(x - 5)^2 + (y + 4)^2 = 100$

6 Coded regression line (AQA, Edexcel, OCR, MEI)

Data pairs (x, y) are coded by $u = x - 6$ and $v = \frac{y}{20}$. The regression line of v on u is $v = 12.5 + 1.2u$.

(a) Find the regression line of y on x in the form $y = a + bx$.
(b) Use it to estimate y when $x = 14$.

Substitute $u = x - 6$ and $v = y/20$: $\frac{y}{20} = 12.5 + 1.2(x - 6)$

$y = 20[12.5 + 1.2x - 7.2] = 20 \times 5.3 + 20 \times 1.2x$
$y = 106 + 24x$
At $x = 14$: $y = 106 + 24 \times 14 = 442$
Answer: (a) $y = 106 + 24x$ (b) 442

7 Reciprocal graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the curve C, its asymptotes (dashed) and a point P. C has equation $y = \frac{a}{x + p} + q$. Find a, p and q.

Asymptotes $x = -3$ and $y = -2$, so $p = 3$ and $q = -2$.

$y = \frac{a}{x + 3} - 2$.

P(−2, 4): $4 = \frac{a}{1} - 2$, so $a = 6$.

Answer: $a = 6$, $p = 3$, $q = -2$

8 Indices: simultaneous equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously, giving integer x and y:

$2^x \times 4^y = 32$

$9^x \div 27^y = 27$

Write each side with prime bases: $32 = 2^5$, $27 = 3^3$.
First: $2^{1x} \times 2^{2y} = 2^5$, so $1x + 2y = 5$.
Second: $3^{2x - 3y} = 3^3$, so $2x - 3y = 3$.
Solving the pair of linear equations: $x = 3$, $y = 1$.
Answer: $x = 3$, $y = 1$

9 Lines in 3D: intersection and angle (CIE (9709); Further Maths on other boards)

Line l_1: $\mathbf{r} = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix} + s\begin{pmatrix} -2 \\ -2 \\ -3 \end{pmatrix}$,

line l_2: $\mathbf{r} = \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$.

Show whether the lines meet, giving the point of intersection if they do, and find the acute angle between them to 1 d.p.

Equate components: $6 - 2s = 4 - 2t$; $3 - 2s = -2 + t$; $5 - 3s = -2 + t$
Solving the first two: $s = 2$, $t = 1$; the third equation is satisfied
Intersection: (2, −1, −1)

$\cos\varphi = \frac{|-1|}{\sqrt{17}\sqrt{6}} = 0.0990$, $\varphi = 84.3°$

Answer: Lines meet at (2, −1, −1); angle 84.3°

10 One-tailed binomial test with data (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A school says that 10% of its pupils walk to school. 40 pupils are surveyed, and 7 walk.
Test, at the 1% significance level, the claim that the proportion is higher.
Give your conclusion in context.

H_0: p = 0.1, H_1: p > 0.1. Let X ~ B(40, 0.1)

$P(X \geq 7) = 1 - P(X \leq 6) = 0.0995$

0.0995 is greater than 1%, so do not reject H_0

There is insufficient evidence that the proportion is higher than the claimed 0.1

Answer: Do not reject H_0: p-value 0.0995 > 1%