Day 324 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Pair of inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the set of values of x for which both

$x^2 - 5x - 6 \geq 0$ and $2x + 4 > -5$.

Quadratic: $(x + 1)(x - 6) \geq 0$ gives $x \leq -1$ or $x \geq 6$.

Linear: $2x + 4 > -5$ gives $x > -\frac{9}{2}$.

Both must hold: take the overlap of the two sets.

Answer: $-\frac{9}{2} < x \leq -1$ or $x \geq 6$

2 Pdf of the form a + bx with a given mean (CIE)

The pdf of X is $f(x) = a + bx$ for $0 \leq x \leq 2$, and 0 otherwise, where a and b are constants. $E(X) = \frac{37}{30}$.

(a) Find a and b.

(b) Find $P(X < 1)$.

Total area = 1: $2a + 2b = 1$

$E(X) = \int x(a + bx)\,dx = 2a + \frac{8}{3}b = \frac{37}{30}$

Solving the simultaneous equations: $a = \frac{3}{20}$, $b = \frac{7}{20}$

$P(X < 1) = \int f(x)\,dx$ from 0 to 1 $= \frac{3}{20} \times 1 + \frac{7}{20} \times 0.5$

$= \frac{13}{40}$

Answer: (a) $a = \frac{3}{20}$, $b = \frac{7}{20}$ (b) $\frac{13}{40}$

3 Logarithmic equations with ln (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $2 \ln x = \ln(x + 72) + \ln 1$,

stating clearly why one root is rejected.

$\ln x^2 = \ln(1(x + 72))$, so $x^2 = 1x + 72$

$x^2 - 1x - 72 = 0$, $(x - 9)(x + 8) = 0$

$x = -8$ is rejected since $\ln x$ is undefined for $x < 0$

Answer: $x = 9$

4 Cubic with only one real root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 + 5x^2 + 11x + 10$.

Show that $(x + 2)$ is a factor of $f(x)$, and that the equation $f(x) = 0$ has exactly one real root.

$f(-2) = -8 + 20 - 22 + 10 = 0$, so $(x + 2)$ is a factor.

$f(x) = (x + 2)(x^2 + 3x + 5)$.

For $x^2 + 3x + 5 = 0$: $b^2 - 4ac = 9 - 20 = -11 < 0$.

No real roots from the quadratic, so the only real root is $x = -2$.

Answer: Proved; the only real root is $x = -2$

5 Identity then equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Show that $\frac{\sin 2\theta}{1 + \cos 2\theta} \equiv \tan\theta$.

(b) Hence solve $\frac{\sin 2\theta}{1 + \cos 2\theta} = 3$ for $0 \leq \theta < 360°$ (1 d.p.).

$\frac{\sin 2\theta}{1 + \cos 2\theta} = \frac{2\sin\theta\cos\theta}{2\cos^2\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$

$\tan\theta = 3$, so $\theta = 71.6°$ (principal value) and add 180°

Answer: $\theta = 71.6°, 251.6°$

6 Polynomial inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $x^3 - 5x^2 - x + 5 < 0$.

Use the factor theorem to find roots: -1, 1, 5.
$f(x) = (x + 1)(x - 1)(x - 5)$.
Sign of f in each region (left to right): $-$, $+$, $-$, $+$.
Take the regions where $f < 0$ (include roots if $\leq$ or $\geq$).
Answer: $x < -1$ or $1 < x < 5$

7 Boat crossing a river (AQA, Edexcel, OCR, MEI)

A river is 80 m wide and flows at 34 m s^{-1}. A boat has speed 16 m s^{-1} in still water and crosses the river heading so that it is carried as little distance downstream as possible.
Find the least possible distance downstream between the starting point and the landing point.

The boat speed is less than the current, so it cannot stop drifting. Let the boat head at angle φ upstream from the perpendicular to the bank

Across: $16\cos\varphi$; along (downstream): $34 - 16\sin\varphi$. Drift = $\frac{80}{16\cos\varphi}(34 - 16\sin\varphi)$

Drift is least when the resultant velocity is at the greatest angle to the downstream direction: the boat velocity is perpendicular to the resultant, so $\sin\varphi = \frac{16}{34}$

Then across speed = $\frac{240}{17}$ and downstream speed = $\frac{450}{17}$, so drift = $80 \times \frac{\frac{450}{17}}{\frac{240}{17}}$

Answer: 150 m

8 Inverse trigonometric values (Edexcel, OCR, MEI, Eduqas, CIE (inverse trig))

Find the exact value of

$\tan(\tan^{-1}(\frac{1}{6}) + \tan^{-1}(\frac{4}{3}))$.

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$= \frac{\frac{1}{6} + \frac{4}{3}}{1 - \frac{1}{6} \times \frac{4}{3}} = \frac{\frac{3}{2}}{\frac{7}{9}}$

Answer: $\frac{27}{14}$

9 Reading a log graph (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$y = ab^x$. The graph of lg y against x is the straight line shown.
Find a and b to 3 significant figures.

$\lg y = \lg a + x \lg b$. The intercept is $\lg a = 1.2$, so $a = 10^{1.2} = 15.85$

Gradient = $\frac{1.6 - 1.2}{4} = 0.1$

$\lg b = 0.1$ so $b = 10^{0.1} = 1.259$
Answer: a = 15.8, b = 1.26

10 Acceleration given, find when P returns to O (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle P moves on a straight line. At time t = 0, P passes through the fixed point O with velocity 80 m s^{-1}. At time t seconds, P has acceleration $(3t - 26)$ m s^{-2}.
Find (a) the times at which P is at O again, (b) the greatest distance of P from O before it first returns to O.

$v = \int a\,dt = 1.5t^2 - 26t + 80$ ($v = 80$ when $t = 0$)
$s = \int v\,dt = 0.5t^3 - 13t^2 + 80t$ ($s = 0$ when $t = 0$)
$s = 0.5t(t - 10)(t - 16)$, so P is at O when $t = 10$ and $t = 16$
Greatest distance when $v = 0$: $1.5t^2 - 26t + 80 = 0$, so $t = 4$ (the smaller root)
$|s| = 0.5 \times 4 \times 6 \times 12$
Answer: (a) t = 10, t = 16 (b) 144 m