Day 323 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Small angle approximations (AQA, Edexcel, OCR, MEI, Eduqas, CIE (small angle))

θ is small and measured in radians. Use small angle approximations to find the value that $\frac{\sin 1\theta \tan 2\theta}{1 - \cos 4\theta}$ approaches as $\theta \to 0$.

$\sin 1\theta \approx 1\theta$, $\tan 2\theta \approx 2\theta$, $1 - \cos 4\theta \approx 8\theta^2$

Expression $\approx \frac{1\theta \times 2\theta}{8\theta^2} = \frac{1}{4}$

Answer: $\mathbf{\frac{1}{4}}$

2 Three-stage journey (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A train accelerates uniformly from rest at 0.6 m s^{-2} to a speed of V m s^{-1}, travels at this speed, then decelerates uniformly at 0.75 m s^{-2} to rest.
The journey takes 88 s and covers 840 m. Find V and the time spent at constant speed.

Time accelerating = $\frac{V}{0.6}$, time decelerating = $\frac{V}{0.75}$, so time at constant speed = $88 - \frac{V}{0.6} - \frac{V}{0.75}$

Area under the speed-time graph: $840 = 88V - \frac{V^2}{1.2} - \frac{V^2}{1.5}$

Multiply by 2ab: $1.35V^2 - 79.2V + 756 = 0$

$V = 12$ or $V = \frac{140}{3}$

$V = \frac{140}{3}$ gives time at constant speed -52 s, which is negative, so $V = 12$

Time at constant speed = $88 - 20 - 16 = 52$ s

Answer: V = 12, 52 s

3 Proof by contradiction (irrationality) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Given that $\sqrt{3}$ is irrational, prove by contradiction that $1 + 8\sqrt{3}$ is irrational.

Assume $1 + 8\sqrt{3} = r$, where r is rational.

Then $\sqrt{3} = \frac{r - 1}{8}$.

$r - 1$ is rational and 8 is a non-zero integer,

so $\frac{r - 1}{8}$ is rational. This contradicts $\sqrt{3}$ irrational.

Hence $1 + 8\sqrt{3}$ is irrational.

Answer: Proved

4 Coded regression line (AQA, Edexcel, OCR, MEI)

Data pairs (x, y) are coded by $u = x - 2$ and $v = \frac{y}{2}$. The regression line of v on u is $v = 37 + 0.75u$.

(a) Find the regression line of y on x in the form $y = a + bx$.
(b) Use it to estimate y when $x = 4$.

Substitute $u = x - 2$ and $v = y/2$: $\frac{y}{2} = 37 + 0.75(x - 2)$

$y = 2[37 + 0.75x - 1.5] = 2 \times 35.5 + 2 \times 0.75x$
$y = 71 + 1.5x$
At $x = 4$: $y = 71 + 1.5 \times 4 = 77$
Answer: (a) y = 71 + 1.5x (b) 77

5 Tangent and axes triangle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The tangent to $y = \frac{x^2 + 2}{x + 1}$ at the point where $x = 1$ meets the axes at A and B. Find the exact area of triangle OAB, where O is the origin.

At $x = 1$: $y = \frac{3}{2}$. Differentiate: gradient $= \frac{1}{4}$

Tangent: $y - \frac{3}{2} = \frac{1}{4}(x - 1)$, so $y = \frac{1}{4}x + \frac{5}{4}$

y-intercept $\frac{5}{4}$; x-intercept -5

Area $= \frac{1}{2} \times |\frac{5}{4}| \times |-5|$

Answer: $\frac{25}{8}$

6 Closest approach of two ships (AQA, Edexcel, OCR, MEI, Eduqas)

Two ships A and B move with constant velocities. At time t = 0 hours, A is at $(-8\mathbf{i} + 7\mathbf{j})$ km relative to a fixed origin and B is at $(-9\mathbf{i})$ km. A has velocity $(-4\mathbf{j})$ km h^{-1} and B has velocity $(4\mathbf{i} - \mathbf{j})$ km h^{-1}.
Find (a) the time at which the ships are closest together, (b) the least distance between them.

Position of B relative to A: $\mathbf{d} = (-\mathbf{i} - 7\mathbf{j}) + t(4\mathbf{i} + 3\mathbf{j})$

$|\mathbf{d}|^2 = (4t - 1)^2 + (3t - 7)^2 = 25t^2 - 50t + 50$

Least when $\frac{d}{dt}|\mathbf{d}|^2 = 0$: $t = -\frac{-25}{25} = 1$

$\mathbf{d} = (-\mathbf{i} - 7\mathbf{j}) + 1(4\mathbf{i} + 3\mathbf{j}) = (3\mathbf{i} - 4\mathbf{j})$, so least distance = 5

Answer: (a) t = 1 h (b) 5 km

7 Partial fractions (repeated factor) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{-33x^2 + 85x - 55}{(x - 4)(3x - 3)^2}$ in the form $\frac{A}{(x - 4)} + \frac{B}{(3x - 3)} + \frac{C}{(3x - 3)^2}$.

$-33x^2 + 85x - 55 \equiv A(3x - 3)^2 + B(x - 4)(3x - 3) + C(x - 4)$.

Put $x = 4$: $-243 = A(9)^2$, so $A = -3$.

Put $x = 1$: $-3 = C(-3)$, so $C = 1$.

Compare x^2: $-33 = 9A + 3B$, so $B = -2$.

Answer: A = −3, B = −2, C = 1

8 Parametric to Cartesian (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations
$x = 2 \sin t$, $y = 3 \cos 2t$.
Find y in terms of x.

$\cos 2t = 1 - 2\sin^2 t$.

$\sin t = \frac{x}{2}$.

$y = 3(1 - 2 \times \frac{x^2}{4}) = 3 - \frac{3}{2}x^2$.

Answer: $y = 3 - \frac{3}{2}x^2$

9 Lines in 3D: intersection and angle (CIE (9709); Further Maths on other boards)

Line l_1: $\mathbf{r} = \begin{pmatrix} 6 \\ -4 \\ -8 \end{pmatrix} + s\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$,

line l_2: $\mathbf{r} = \begin{pmatrix} 5 \\ 8 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$.

Show whether the lines meet, giving the point of intersection if they do, and find the acute angle between them to 1 d.p.

Equate components: $6 - s = 5 + 2t$; $-4 + 3s = 8 + 3t$; $-8 + 2s = 0$

First two give $s = 3$, $t = -1$, but these fail in the third equation, so the lines do not meet (skew)

$\cos\varphi = \frac{|7|}{\sqrt{14}\sqrt{13}} = 0.5189$, $\varphi = 58.7°$

Answer: Lines do not meet (skew); angle 58.7°

10 Normal test: critical region and Type II error (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim N(\mu, 16)$. H_0: $\mu = 46$ is tested against H_1: $\mu > 46$ using the mean of a sample of size 9 at the 1% level.

(a) Find the critical region for the sample mean.

(b) Find the probability of a Type II error when $\mu = 48$.

Under H_0, sample mean $\sim N(46, \frac{16}{9})$, standard error = 1.333

$z = 2.3263$ for the upper tail: critical value = $46 + 2.3263 \times 1.333 = 49.10$

Type II error: do not reject H_0, sample mean < 49.10, where mean ~ N(48, 1.333)

$z = \frac{49.10 - 48}{1.333} = 0.826$

Probability = 0.796

Answer: (a) sample mean > 49.1 (b) 0.796