Day 322 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Equations in disguise (substitution) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $x - 5\sqrt{x} + 6 = 0$.

Let $u = \sqrt{x}$: $u^2 - 5u + 6 = 0$.
$(u - 2)(u - 3) = 0$, so $\sqrt{x} = 2$ or $\sqrt{x} = 3$.
Square: $x = 4$ or $x = 9$.
Answer: $x = 4$ or $x = 9$

2 Parametric differentiation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = \ln t$, $y = 2t^2 + 3t$, $t > 0$.
Find the equation of the tangent where $t = 1$, in the form $y = mx + c$.

$dx/dt = \frac{1}{t}$, $dy/dt = 4t + 3$
$dy/dx = t(4t + 3)$ which is 7 at $t = 1$
At $t = 1$: $x = 0$, $y = 5$
Answer: $y = 7x + 5$

3 Velocity-time graph with a return (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the velocity-time graph of a particle P moving on a straight line (velocity in m s^{-1}, time in s). At time 15 s, P is back at its starting point.
Find (a) the value of W, (b) the greatest distance of P from its starting point.

Net area = 0 (back at start): $\frac{1}{2} \times 2 \times 6 + \frac{1}{2}(6 - W)(5) - W(8) = 0$
$6 + 15 - 2.5W - 8W = 0$, so $10.5W = 21$
$W = 2$

$v = 0$ where the line from (2, 6) to (7, −2) crosses the axis: $t = 2 + 5 \times \frac{6}{8} = 5.75$

Greatest distance = area of triangle = $\frac{1}{2} \times 5.75 \times 6$
Answer: (a) $W = 2$ (b) 17.25 m

4 Least number of trials (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each component is faulty with probability 0.12, independently.
Find the least number n of components that must be tested so that the probability of finding at least two faulty is greater than 0.9.

Let $X \sim B(n, 0.12)$. $P(X \geq 2) = 1 - 0.88^n - n(0.12)(0.88)^{n-1}$
This has no simple algebraic solution, so test values of n
$n = 30$: $P(X \geq 2) = 0.8900$, which is not greater than 0.9
$n = 31$: $P(X \geq 2) = 0.9006$, which is greater than 0.9
$P(X \geq 2)$ increases with n, so the least value is 31
Answer: $n = 31$

5 Integration by substitution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use the substitution $u = e^x + 3$ to find the exact value of

$$\int_0^{\ln 7} \frac{e^x}{e^x + 3}\, dx$$

$du = e^x\, dx$; limits $x = 0$, $\ln 7$ become $u = 4$, 10

Integral = $\int_4^{10} \frac{1}{u}\, du = [\ln u]$

Evaluate the limits
Answer: $-\ln 2 + \ln 5$

6 Proof by exhaustion (modular) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove that $3k = n^2 + 1$ has no integer solutions (k and n integers).

Every integer is 3q, 3q + 1, 3q + 2 for some integer q.

Squaring, $n^2 \equiv r^2 \pmod 3$, so:

$0 \to 0$; $1 \to 1$; $2 \to 1$ ($r \to n^2 \bmod 3$)

So $n^2 + 1 \equiv 1$ or $2 \pmod 3$.

No residue is 0, so $n^2 + 1$ is never a multiple of 3.

Answer: Proved

7 Car on a hill: power and maximum speed (CIE)

A car of mass 1200 kg, with engine power 31 kW, climbs a straight road with $\sin\alpha = \frac{1}{14}$ against a constant resistance of 400 N.

Find (a) the greatest steady speed up the hill, (b) the acceleration at 6.25 m s^{-1}. Take g = 9.8 m s^{-2}.

Steady speed: driving force = $400 + 1200 \times 9.8 \times \frac{1}{14} = 1240$ N

$P = F v$, so $v = \frac{31000}{1240} = 25$

At 6.25 m s^{-1}: $F = \frac{31000}{6.25} = 4960$ N

$F - 1240 = 1200\,a$, so $a = \frac{3720}{1200}$

Answer: (a) 25 m s^{-1} (b) 3.1 m s^{-2}

8 Implicit differentiation: tangent or normal (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $3x^2 - 4xy + 3y^2 = 10$.

Find the equation of the tangent to C at the point (−1, 1),

in the form ax + by + c = 0 with integers a, b, c.

Differentiate implicitly with respect to x

At (−1, 1): dy/dx = 1

$y - 1 = 1(x - -1)$, then clear fractions

Answer: x − y + 2 = 0

9 Tangent line to circle: finding k (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The line $5y = 12x + k$ is a tangent to the circle

$(x - 2)^2 + (y - 4)^2 = 16$.

Find the two possible values of k.

Substitute $y = \frac{12x + k}{5}$: $(x - 2)^2 + (\frac{12x + k}{5} - 4)^2 = 16$.

Multiply by 25 and collect: a quadratic in x with a = 169.

Tangent ⇒ discriminant = 0. This simplifies to $(k + 4)^2 = 2704$.

$k + 4 = \pm 52$.

Answer: k = −56 or k = 48

10 Most likely value of a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(18, \frac{1}{4})$.

(a) Show that $P(X = r + 1) \div P(X = r) = \frac{18 - r}{r + 1} \times \frac{1}{3}$.

(b) Find the most likely value of X and its probability (3 s.f.).

$P(X = r + 1) \div P(X = r) = \frac{C(18, r + 1)}{C(18, r)} \times \frac{p}{q} = \frac{18 - r}{r + 1} \times \frac{p}{q}$

With $p = \frac{1}{4}$ and $q = \frac{3}{4}$, $p \div q = \frac{1}{3}$

The ratio exceeds 1 when $(18 - r)\frac{1}{4} > (r + 1)\frac{3}{4}$, that is $r < 3.75$

So probabilities rise up to r = 4 and then fall; the mode is 4

$P(X = 4) = 0.213$

Answer: 4 with probability 0.213