Day 320 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Trig equations with compound angles (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\operatorname{cosec}(3\theta + 45°) = \sqrt{2}$
for $0° \le \theta \le 180°$.

Let $u = 3\theta + 45°$; then $\operatorname{cosec} u = \sqrt{2}$
Base solutions: $u = 135°, 45°$ (+ multiples of 360°)
$\theta = (u - 45°) \div 3$, keeping those in range
Answer: $\theta = 0°, 30°, 120°, 150°$

2 Tangent line to circle: finding k (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The line $8y = 6x + k$ is a tangent to the circle
$(x + 3)^2 + (y + 3)^2 = 4$.
Find the two possible values of k.

Substitute $y = \frac{6x + k}{8}$: $(x + 3)^2 + (\frac{6x + k}{8} + 3)^2 = 4$.
Multiply by 64 and collect: a quadratic in x with $a = 100$.
Tangent $\Rightarrow$ discriminant = 0. This simplifies to $(k + 6)^2 = 400$.
$k + 6 = \pm 20$.
Answer: $k = -26$ or $k = 14$

3 Normal: unknown standard deviation then a second probability (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The lengths of a component, in mm are modelled by $N(92, \sigma^2)$. The probability of a value below 87 is 0.30.
(a) Find σ.
(b) Find the probability of a value below 79.

$P(X < 87) = 0.30$ so $Z = \frac{87 - 92}{\sigma} = -0.5244$

$\sigma = \frac{-5}{-0.5244} = 9.535$

$z = \frac{79 - 92}{9.535} = -1.363$

$P(X < 79) = P(Z < -1.363) = 0.0864$
Answer: (a) 9.53 (b) 0.0864

4 Fixed point iteration and rearrangement (AQA, Edexcel, OCR, MEI, Eduqas, CIE (iterative methods))

The equation $x^3 = 5x + 7$ has a root α.
(a) Show that the equation can be written as $x = \sqrt[3]{(5x + 7)}$.
(b) Use $x_{n+1} = \sqrt[3]{(5x_n + 7)}$ with $x_0 = 2$ to find x_1, x_2, x_3
to 4 d.p., and hence state α to 3 d.p.

Take the cube root of both sides to obtain $x = \sqrt[3]{(5x + 7)}$
$x_1 = 2.5713$, $x_2 = 2.7079$, $x_3 = 2.7386$
The iteration converges (successive values agree to 3 d.p.): $\alpha = 2.747$
Answer: $x_1 = 2.5713$, $x_2 = 2.7079$, $x_3 = 2.7386$; $\alpha = 2.747$

5 Rational inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\frac{-3x + 1}{2x + 2} \ge -2$.

Rearrange: $\frac{-3x + 1}{2x + 2} - (-2) \ge 0$.

Combine: $\frac{x + 5}{2x + 2} \ge 0$.

Critical values: $x = -5$ (zero of numerator), $x = -1$ (excluded, denominator zero).
Sign analysis of the three regions gives the solution.
Answer: $x \le -5$ or $x > -1$

6 sec, cosec and tan/cot quadratics (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $12\text{cosec}^2\theta + 14\cot\theta - 8 = 0$
for $0 \leq \theta < 360°$, giving answers to 1 d.p.

Use $\text{cosec}^2\theta \equiv 1 + \cot^2\theta$: $12\cot^2\theta + 14\cot\theta + 4 = 0$
$2(2\cot\theta + 1)(3\cot\theta + 2) = 0$
$\cot\theta = -\frac{1}{2}$ or $\cot\theta = -\frac{2}{3}$
$\tan\theta = -2$ or $-\frac{3}{2}$, so $\theta = 116.6°$ and $123.7°$ $(+180°)$
Answer: θ = 116.6°, 123.7°, 296.6°, 303.7°

7 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

A shell is projected from a point O on horizontal ground with speed 24.5 m s^{-1} at an angle θ above the horizontal, where $\tan\theta = \frac{3}{4}$. Find the horizontal distance between the two points at which the shell is 6.125 m above the ground. Take g = 9.8 m s^{-2}.

Vertical: $6.125 = 14.7t - 4.9t^2$, i.e. $4.9t^2 - 14.7t + 6.125 = 0$
$t = \frac{14.7 \pm \sqrt{(14.7^2 - 19.6 \times 6.125)}}{9.8} = \frac{14.7 \pm 9.8}{9.8}$
The two times differ by 2 s (symmetry about the top of the path)
Horizontal distance = 19.6 × 2
Answer: 39.2 m

8 Displacement of the form R sin(ωt + φ) (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves on a line. Its displacement from a fixed point O at time t seconds is s metres, where $s = 3\sin t + 4\cos t$.
Find (a) the greatest distance of P from O, (b) the greatest speed of P, (c) the first time P is at its greatest distance from O (3 s.f.).

Write $s = R\sin(t + \varphi)$ with $R\cos\varphi = 3$ and $R\sin\varphi = 4$
$R = \sqrt{(3^2 + 4^2)} = 5$ and $\tan\varphi = \frac{4}{3}$, so $\varphi = 0.9273$ rad
Greatest distance = R = 5
$v = \frac{ds}{dt} = 5 \times 1\cos(t + \varphi)$, so greatest speed = 5
s is greatest when $t + \varphi = \frac{\pi}{2}$: $t = (1.5708 - 0.9273) \div 1$
Answer: (a) 5 m (b) 5 m s^{-1} (c) 0.644 s

9 Critical region, significance level and Type II error (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A test of H_0: p = 0.1 against H_1: p > 0.1 uses a sample of 30 at the 5% level.
(a) Find the critical region for X, the number of successes.
(b) Find the actual significance level.
(c) Find P(Type II error) when in fact p = 0.3.

$X \sim B(30, 0.1)$. $P(X \geq 7) = 0.0258$ and $P(X \geq 6) = 0.0732$
The first is at most 0.05 and the second is not, so the critical region is $X \geq 7$
Actual significance level = $P(X \geq 7) = 0.0258$
Type II error: H_0 is not rejected, so $X \leq 6$, with $X \sim B(30, 0.3)$
$P(X \leq 6) = 0.160$
Answer: (a) X ≥ 7 (b) 0.0258 (c) 0.160

10 Recurrence: limit and convergence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = -\frac{1}{3}u_n + 4$, $u_1 = 13$.

(a) Find the limit L of the sequence.

(b) Find the least n such that $|u_n - L| < 0.001$.

(a) At the limit $L = -\frac{1}{3}L + 4$, so $L(1 - (-\frac{1}{3})) = 4$.

$L = 3$.

(b) $u_n - L = (u_1 - L)((-\frac{1}{3}))^{n-1} = 10((-\frac{1}{3}))^{n-1}$.

Need $10 \times (\frac{1}{3})^{n-1} < 0.001$, so $n - 1 > 8.384$.

The least n is 10.

Answer: (a) L = 3 (b) n = 10