Day 32 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Trig equations with compound angles (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\cos(2\theta + 15°) = -\frac{\sqrt{2}}{2}$
for $0° \leq \theta \leq 180°$.

Let $u = 2\theta + 15°$; then $\cos u = -\frac{\sqrt{2}}{2}$
Base solutions: $u = 135, -135°$ (+ multiples of 360°)
$\theta = (u - 15°) \div 2$, keeping those in range
Answer: θ = 60°, 105°

2 Line tangent to a hyperbola (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $y = \frac{-2x + 20}{x - 2}$.
(a) Write down the equations of the asymptotes of C.
(b) The line $y = -x + k$ is a tangent to C.
Find the possible values of k.

(a) Vertical: $x = 2$. As $x \to \infty$, $y \to -2$, so $y = -2$.
(b) $-2x + 20 = (-x + k)(x - 2)$.
$-x^2 + (k + 4)x + (-2k - 20) = 0$.
Tangent: $b^2 - 4ac = 0$, which simplifies to $k^2 - 64 = 0$.
Answer: (a) x = 2, y = −2 (b) k = −8 or k = 8

3 Triangular pdf from a diagram (CIE)

The diagram shows the pdf of X.
Find (a) k, (b) E(X), (c) the median of X (3 s.f.).

(a) Area of triangle = ½ × 12 × k = 1, so $k = \frac{1}{6}$
(b) The mean of a triangular distribution is $\frac{0 + 7 + 12}{3} = \frac{19}{3}$
$F(7) = \frac{7}{12} = \frac{7}{12} \geq \frac{1}{2}$, so the median is in the left part
$F(x) = \frac{x^2}{84} = \frac{1}{2}$, so $x = \sqrt{42} = 6.481$
Answer: (a) $k = \frac{1}{6}$ (b) $E(X) = \frac{19}{3}$ (c) 6.48

4 Parametric curve and line intersection (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has parametric equations
$x = 3t$, $y = t^2$. The line $y = -\frac{2}{3}x + 3$
meets C at A and B. Find the exact length AB.

Substitute: $t^2 = -2t + 3$.
$t^2 + 2t - 3 = 0 \Rightarrow t = 1$ or $t = -3$.
A(3, 1), B(−9, 9).
$AB^2 = 12^2 + (-8)^2 = 208$.
Answer: $AB = 4\sqrt{13}$

5 Trigonometric integrals with identities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use a suitable trigonometric identity to find the exact value of
$\int_0^{\pi/4} \cos^2 2x \, dx$

Use the double angle formula to write the integrand without squares
Integrate term by term
Substitute the limits
Answer: $\frac{1}{8}\pi$

6 Displacement of the form R sin(ωt + φ) (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves on a line. Its displacement from a fixed point O at time t seconds is s metres, where $s = 3\sin\frac{t}{2} + 4\cos\frac{t}{2}$.

Find (a) the greatest distance of P from O, (b) the greatest speed of P, (c) the first time P is at its greatest distance from O (3 s.f.).

Write $s = R\sin(\frac{t}{2} + \varphi)$ with $R\cos\varphi = 3$ and $R\sin\varphi = 4$

$R = \sqrt{(3^2 + 4^2)} = 5$ and $\tan\varphi = \frac{4}{3}$, so $\varphi = 0.9273$ rad

Greatest distance = R = 5

$v = \frac{ds}{dt} = 5 \times \frac{1}{2}\cos(\frac{t}{2} + \varphi)$, so greatest speed = $\frac{5}{2}$

s is greatest when $\frac{t}{2} + \varphi = \frac{\pi}{2}$: $t = (1.5708 - 0.9273) \div \frac{1}{2}$

Answer: (a) 5 m (b) $\frac{5}{2}$ m s^{-1} (c) 1.29 s

7 Logarithmic equation, reject a root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\log_3(x + 20) + \log_3(x - 6) = 3$,
justifying clearly any root that you reject.

$(x + 20)(x - 6) = 3^3 = 27$

$x^2 + 14x - 147 = 0$

$(x - 7)(x + 21) = 0$, so $x = 7$ or $x = -21$

$x = -21$ makes $x + 20 = -1 < 0$ and log of a negative number is undefined

Answer: x = 7

8 Critical region, significance level and Type II error (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A test of H_0: $p = 0.2$ against H_1: $p < 0.2$ uses a sample of 50 at the 1% level.

(a) Find the critical region for X, the number of successes.

(b) Find the actual significance level.

(c) Find P(Type II error) when in fact $p = 0.05$.

$X \sim B(50, 0.2)$. $P(X \leq 3) = 0.00566$ and $P(X \leq 4) = 0.0185$

The first is at most 0.01 and the second is not, so the critical region is $X \leq 3$

Actual significance level = $P(X \leq 3) = 0.00566$

Type II error: H_0 is not rejected, so $X \geq 4$, with $X \sim B(50, 0.05)$

$P(X \geq 4) = 1 - P(X \leq 3) = 0.240$

Answer: (a) X ≤ 3 (b) 0.00566 (c) 0.240

9 Speed decaying exponentially (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves on a straight line. The velocity of P is v m s^{-1} and its acceleration is $-0.25v^2$ m s^{-2}. At t = 0, v = 20.

Find (a) the time taken for the speed of P to halve, (b) the distance P travels in this time, to 3 s.f.

$\frac{dv}{dt} = -0.25v^2$: $\int\frac{dv}{v^2} = -0.25\int dt$, so $\frac{1}{v} = 0.25t + c$, and $\frac{1}{v} = 0.25t + \frac{1}{20}$

$v = 10$: $\frac{1}{v} = 0.1$, so $0.25t = 0.1 - 0.05$, giving $t = 0.2$

$s = \int\frac{1}{(0.25t + 1/20)}\,dt = \frac{1}{0.25}\ln(1 + 0.25 \times 20t)$

At t = 0.2: $s = \frac{1}{0.25}\ln 2$

Answer: (a) 0.2 s (b) 2.77 m

10 Circle touching the axes (diagram) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The circle in the diagram touches both axes and passes through P. Find the equation of the circle, giving it in the form $x^2 + y^2 + ax + by + c = 0$.

Touches both axes in the first quadrant: centre (r, r), radius r.

P(2, 4) lies on the circle: $(2 - r)^2 + (4 - r)^2 = r^2$.

$r^2 - 12r + 20 = 0$, so r = 2 or r = 10.

The diagram shows P on the larger circle, so r = 10.

$(x - 10)^2 + (y - 10)^2 = 100$, then expand.

Answer: $x^2 + y^2 - 20x - 20y + 100 = 0$