Day 318 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Cubic with only one real root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 + x^2 - 5x - 21$.
Show that $(x - 3)$ is a factor of $f(x)$, and that the equation $f(x) = 0$ has exactly one real root.

$f(3) = 27 + 9 - 15 - 21 = 0$, so $(x - 3)$ is a factor.
$f(x) = (x - 3)(x^2 + 4x + 7)$.
For $x^2 + 4x + 7 = 0$: $b^2 - 4ac = 16 - 28 = -12 < 0$.
No real roots from the quadratic, so the only real root is $x = 3$.
Answer: Proved; the only real root is $x = 3$

2 Three-stage journey (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A train accelerates uniformly from rest at 1.5 m s^{-2} to a speed of V m s^{-1}, travels at this speed, then decelerates uniformly at 1.5 m s^{-2} to rest.
The journey takes 74 s and covers 792 m. Find V and the time spent at constant speed.

Time accelerating = $\frac{V}{1.5}$, time decelerating = $\frac{V}{1.5}$, so time at constant speed = $74 - \frac{V}{1.5} - \frac{V}{1.5}$

Area under the speed-time graph: $792 = 74V - \frac{V^2}{3} - \frac{V^2}{3}$

Multiply by 2ab: $3V^2 - 333V + 3564 = 0$
$V = 12$ or $V = 99$
$V = 99$ gives time at constant speed -58 s, which is negative, so $V = 12$
Time at constant speed = $74 - 8 - 8 = 58$ s
Answer: V = 12, 58 s

3 Self-inverse function (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{kx + 9}{x - 6}$ for $x \neq 6$. Given that f is self-inverse, find k, and hence solve $f(x) = x$.

Let $y = f(x)$: $y(x - 6) = kx + 9$, so $x(y - k) = 6y + 9$.
$f^{-1}(x) = \frac{6x + 9}{x - k}$.
$f = f^{-1}$ requires $k = 6$.
$f(x) = x$: $6x + 9 = x(x - 6)$, so $x^2 - 12x - 9 = 0$.
$x = 6 \pm \sqrt{45} = 6 \pm 3\sqrt{5}$.
Answer: $k = 6$; $x = 6 \pm 3\sqrt{5}$

4 Fixed point iteration and rearrangement (AQA, Edexcel, OCR, MEI, Eduqas, CIE (iterative methods))

The equation $x^3 = x + 10$ has a root α.
(a) Show that the equation can be written as $x = \sqrt[3]{(x + 10)}$.
(b) Use $x_{n+1} = \sqrt[3]{(x_n + 10)}$ with $x_0 = 1$ to find x_1, x_2, x_3 to 4 d.p., and hence state α to 3 d.p.

Take the cube root of both sides to obtain $x = \sqrt[3]{(x + 10)}$
$x_1 = 2.2240$, $x_2 = 2.3036$, $x_3 = 2.3086$
The iteration converges (successive values agree to 3 d.p.): $\alpha = 2.309$
Answer: $x_1 = 2.2240$, $x_2 = 2.3036$, $x_3 = 2.3086$; $\alpha = 2.309$

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5 Distribution of the larger of two scores (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Two fair 10-sided dice, numbered 1 to 10, are rolled. M is the larger of the two scores.

(a) Show that $P(M = m) = \frac{(2m - 1)}{100}$ for m = 1, 2, ..., 10.

(b) Find P(M = 3) and E(M).

(a) $M \leq m$ means both scores $\leq m$: $P(M \leq m) = (m/10)^2$, so $P(M = m) = \frac{m^2 - (m - 1)^2}{100} = \frac{(2m - 1)}{100}$

(b) $P(M = 3) = \frac{1}{20}$

$E(M) = \Sigma\, m\, P(M = m) = \frac{1}{100} \Sigma\, m(2m - 1)$

$= \frac{143}{20}$

Answer: (b) $P(M = 3) = \frac{1}{20}$, $E(M) = \frac{143}{20}$

6 Parametric differentiation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = t^2$, $y = t^3 + 24t$.
Find the coordinates of the points on the curve where the gradient is 9.

$dx/dt = 2t$, $dy/dt = 3t^2 + 24$, so $dy/dx = \frac{3t^2 + 24}{2t}$

Set equal to 9: $3t^2 - 18t + 24 = 0$
t = 2 or t = 4
Answer: (4, 56) and (16, 160)

7 Expected winnings and a fair game (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A bag has 3 red, 6 blue and 2 green counters. A player takes two without replacement. Two reds win £3, two blues win £8, anything else wins nothing.
(a) Find the expected winnings.
(b) The player pays £2 to play. Find the expected profit.

$P(RR) = \frac{3}{11} \times \frac{2}{10} = \frac{3}{55}$

$P(BB) = \frac{6}{11} \times \frac{5}{10} = \frac{3}{11}$

(a) $E = 3 \times \frac{3}{55} + 8 \times \frac{3}{11} = \frac{129}{55}$

(b) Expected profit $= \frac{129}{55} - 2 = \frac{19}{55}$

Answer: (a) £2.35 (b) £0.345

8 Range and domain by completing the square (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{5}{x^2 - 4x + 10}$ for all real x.

Find the greatest value of f(x), the value of x at which it occurs, and the range of f.

$x^2 - 4x + 10 = (x - 2)^2 + 6$.

$(x - 2)^2 \geq 0$, so the denominator is at least 6, with equality at x = 2.
The denominator is always positive, so f(x) > 0.

Greatest $f = \frac{5}{6} = \frac{5}{6}$ at x = 2.

Answer: greatest value $\frac{5}{6}$ at x = 2; range $0 < f(x) \leq \frac{5}{6}$

9 Hinged rod supported by a string (AQA, Edexcel, OCR, MEI)

A uniform rod AB (3 m, 40 N) is hinged at A to a vertical wall and held horizontal by a string BC, C on the wall above A, with $\tan\alpha = \frac{5}{12}$, where α is the angle between string and rod. A 50 N particle hangs at P, AP = 1.5 m.

Find (a) the tension, (b) the magnitude and direction of the hinge force (3 s.f. and 1 d.p.).

$\sin\alpha = \frac{5}{13}$, $\cos\alpha = \frac{12}{13}$. Moments about A: $T\sin\alpha \times 3 = 40 \times 1.5 + 50 \times 1.5$

$T \times \frac{5}{13} \times 3 = 135$, so $T = 117$

Horizontal: $H = T\cos\alpha = 108$. Vertical: $V + T\sin\alpha = 40 + 50$, so $V = 45$

Force $= \sqrt{(108^2 + 45^2)} = 117$, angle $= \tan^{-1}(\frac{45}{108}) = 22.6°$ above the horizontal

Answer: (a) 117 N (b) 117 N at 22.6° above the horizontal (away from the wall)

10 Separable differential equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A population N satisfies $\frac{dN}{dt} = -0.005N^2$, with N = 50 when t = 0.

Find N when t = 12, and the time taken for N to halve.

$\int N^{-2}\,dN = -0.005\int dt$ gives $-\frac{1}{N} = -0.005t + c$

t = 0: $c = -\frac{1}{50}$, so $\frac{1}{N} = 0.005t + \frac{1}{50}$

t = 12: $\frac{1}{N} = \frac{3}{50} + \frac{1}{50}$; halving: $0.005t = \frac{1}{50}$

Answer: $N = \frac{25}{2}$; $t = 4$