Day 317 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Trig equations with compound angles (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\sec(3\theta + 25°) = -\frac{2}{\sqrt{3}}$ for $-90° \le \theta \le 270°$.

Let $u = 3\theta + 25°$; then $\sec u = -\frac{2}{\sqrt{3}}$

Base solutions: $u = 150, -150°$ (+ multiples of 360°)

$\theta = (u - 25°) \div 3$, keeping those in range

Answer: $\theta = -78.3°, -58.3°, 41.7°, 61.7°, 161.7°, 181.7°$

2 Continuous random variable: pdf with unknown constant (CIE)

The pdf of X is $f(x) = kx(3 - x)$ for $0 \le x \le 3$, and 0 otherwise.

(a) Find k.

(b) Find E(X) and Var(X).

(c) Find P(X > 0.75).

(a) $\int kx(3 - x)\,dx$ from 0 to 3 $= k[1.5x^2 - x^3/3] = k \times 4.5 = 1$

$k = \frac{2}{9}$

(b) $E(X) = \int x f(x)\,dx = \frac{3}{2}$ by symmetry about $x = \frac{3}{2}$

$E(X^2) = \int x^2 f(x)\,dx = \frac{27}{10}$, so $\text{Var}(X) = \frac{27}{10} - \frac{3}{2}^2 = \frac{9}{20}$

(c) $P(X > 0.75) = \int f(x)\,dx$ from 0.75 to 3 $= \frac{27}{32}$

Answer: (a) $k = \frac{2}{9}$ (b) $E(X) = \frac{3}{2}$, $\text{Var}(X) = \frac{9}{20}$ (c) $\frac{27}{32}$

3 Periodic recurrence relations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sequence is defined by $u_1 = 7$ and

$u_{n+1} = \frac{u_n - 1}{3u_n - 1}$.

Show that the sequence is periodic and find $\sum_{r=1}^{100} u_r$.

$u_1 = 7$, $u_2 = \frac{3}{10}$, $u_3 = 7$.

$u_3 = u_1$, so the sequence repeats with period 2.

One period sums to $\frac{73}{10}$.

$100 = 2 \times 50 + 0$: 50 full periods plus the first 0 terms.

Answer: 365

4 Rough table and smooth slope: string cut (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particle A (2 kg) lies on a rough table with $\mu = 0.25$, and is joined by a light string over a smooth pulley (see diagram) to particle B (1 kg) on a smooth slope, $\sin\alpha = \frac{4}{5}$.

Released from rest, the string is cut when B has moved 2.5 m. Find the further distance A slides before stopping.

A: $T - 0.25 \times 2 \times 9.8 = 2a$. B: $1 \times 9.8 \times \frac{4}{5} - T = 1a$

Adding: $9.8(1 \times \frac{4}{5} - 0.25 \times 2) = 3a$, so $a = 0.98$

$T = 2(0.98 + 0.25 \times 9.8) = 6.86$

When cut: $v^2 = 2 \times 0.98 \times 2.5 = 4.9$. Then A decelerates at $\mu g = 2.45$

Further distance $= \frac{4.9}{2 \times 2.45} = 1$

Answer: 1 m

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5 Transformations of y = x^2 (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Describe a sequence of two transformations that maps the curve $y = x^2$ onto the curve $y = -3x^2 - 12x - 7$.

Complete the square: $y = -3(x + 2)^2 + 5$.

Step 1: $y = x^2 \rightarrow y = -3x^2$: a stretch, scale factor 3, parallel to the y-axis and a reflection in the x-axis.

Step 2: replace x by (x + 2) and add 5: translation by $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$.

Answer: a stretch, scale factor 3, parallel to the y-axis and a reflection in the x-axis, then translation $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$

6 Histogram: frequency density (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The histogram shows the times, t seconds, of some runners.

Find (a) the total number of runners, (b) an estimate of the number with $5 \leq t < 35$.

Frequency = frequency density × width. Bar 24 to 34: height 6 squares, width 10, frequency 120

So one square of frequency density = 2 runners per second

Class frequencies = 40, 48, 96, 120, 56

Total = 360

Estimate = 2 × [3(7) + 4(12) + 6(10) + 7(1)] = 272

Answer: (a) 360 (b) 272

7 Parametric differentiation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = t^2$, $y = 2t^3 + 12t$.

Find the coordinates of the points on the curve where the gradient is -9.

$dx/dt = 2t$, $dy/dt = 6t^2 + 12$, so $dy/dx = \frac{6t^2 + 12}{2t}$

Set equal to -9: $6t^2 + 18t + 12 = 0$

$t = -2$ or $t = -1$

Answer: (4, −40) and (1, −14)

8 Acceleration given, find when P returns to O (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle P moves on a straight line. At time t = 0, P passes through the fixed point O with velocity -60 m s^{-1}. At time t seconds, P has acceleration $(-3t + 23)$ m s^{-2}.

Find (a) the times at which P is at O again, (b) the greatest distance of P from O before it first returns to O.

$v = \int a\,dt = -1.5t^2 + 23t - 60$ ($v = -60$ when $t = 0$)

$s = \int v\,dt = -0.5t^3 + 11.5t^2 - 60t$ ($s = 0$ when $t = 0$)

$s = -0.5t(t - 8)(t - 15)$, so P is at O when $t = 8$ and $t = 15$

Greatest distance when $v = 0$: $-1.5t^2 + 23t - 60 = 0$, so $t = \frac{10}{3}$ (the smaller root)

$|s| = 0.5 \times \frac{10}{3} \times \frac{14}{3} \times \frac{35}{3}$

Answer: (a) t = 8, t = 15 (b) $\frac{2450}{27}$ (= 90.7) m

9 Perpendicular from a point to a line (3D) (CIE (9709); Further Maths on other boards)

The line l has equation $\mathbf{r} = \begin{pmatrix} 4 \\ -6 \\ 4 \end{pmatrix} + t\begin{pmatrix} -3 \\ 1 \\ -3 \end{pmatrix}$.

The point P has coordinates $(-1, -4, -3)$.

Find the coordinates of the foot F of the perpendicular from P to l, and the exact distance PF.

F = (a + td) for some t, so $\mathbf{FP} = \begin{pmatrix} -5 \\ 2 \\ -7 \end{pmatrix} - t\begin{pmatrix} -3 \\ 1 \\ -3 \end{pmatrix}$

FP · d = 0: 38 − t(19) = 0, so t = 2

F = (−2, −4, −2)

$\mathbf{PF} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, so PF = $\sqrt{2}$

Answer: F = (−2, −4, −2), PF = $\sqrt{2}$

10 Circle through three points (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A circle passes through A(−8, 13), B(4, −3)
and C(−10, −1). Find its equation.

Centre is on the perpendicular bisector of each chord.

Bisector of AB: 12x − 16y = −104.

Bisector of BC: −14x + 2y = 38.

Solving: centre (−2, 5).

$r^2 = (-8 - (-2))^2 + (13 - 5)^2 = 100$.

Answer: $(x + 2)^2 + (y - 5)^2 = 100$