Day 315 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground with speed 63.7 m s^{-1} at an angle θ above the horizontal, where $\tan\theta = \frac{5}{12}$. Find the horizontal distance between the two points at which the ball is 11.025 m above the ground. Take g = 9.8 m s^{-2}.

Vertical: $11.025 = 24.5\,t - 4.9t^2$, i.e. $4.9t^2 - 24.5t + 11.025 = 0$

$t = \frac{24.5 \pm \sqrt{(24.5^2 - 19.6 \times 11.025)}}{9.8} = \frac{24.5 \pm 19.6}{9.8}$

The two times differ by 4 s (symmetry about the top of the path)

Horizontal distance = 58.8 × 4

Answer: 235.2 m

2 Trig equations with compound angles (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\cos(3\theta + 30°) = -\frac{\sqrt{2}}{2}$

for $-180° \le \theta \le 180°$.

Let $u = 3\theta + 30°$; then $\cos u = -\frac{\sqrt{2}}{2}$

Base solutions: u = 135, −135° (+ multiples of 360°)

$\theta = (u - 30°) \div 3$, keeping those in range

Answer: θ = −-175°, −-85°, −-55°, 35°, 65°, 155°

3 Conditional probability from given values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

For events A and B, P(A) = 0.85, P(B) = 0.2 and P(A | B) = 0.9.

(a) Find P(A ∩ B).

(b) Find P(A′ | B′).

$P(A \cap B) = P(A \mid B) \times P(B) = 0.9 \times 0.2 = 0.18$

$P(A \cup B) = 0.85 + 0.2 - 0.18 = 0.87$

P(B′) = 0.8

$P(A' \cap B') = 1 - P(A \cup B) = 0.13$

$P(A' \mid B') = \frac{0.13}{0.8}$

Answer: (a) 0.18 (b) $\frac{13}{80}$

4 Cubic with only one real root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 + 4x^2 + 12x + 16$.

Show that (x + 2) is a factor of f(x), and that the equation f(x) = 0 has exactly one real root.

$f(-2) = -8 + 16 - 24 + 16 = 0$, so (x + 2) is a factor.

$f(x) = (x + 2)(x^2 + 2x + 8)$.

For $x^2 + 2x + 8 = 0$: $b^2 - 4ac = 4 - 32 = -28 < 0$.

No real roots from the quadratic, so the only real root is x = −2.

Answer: Proved; the only real root is x = −2

5 Identity then equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Show that $\tan\theta + \cot\theta \equiv 2\,\mathrm{cosec}\,2\theta$.

(b) Hence solve $\tan\theta + \cot\theta = 10$ for $0 \le \theta < 360°$ (1 d.p.).

$\tan\theta + \cot\theta = \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} = \frac{1}{\sin\theta\cos\theta} = \frac{2}{\sin 2\theta} = 2\,\mathrm{cosec}\,2\theta$

$2\,\mathrm{cosec}\,2\theta = 10$, so $\sin 2\theta = \frac{2}{10}$, 2θ = 11.5°, 168.5°, 371.5°, 528.5°

Halve each value

Answer: θ = 5.8°, 84.2°, 185.8°, 264.2°

6 Interception at constant speed (AQA, Edexcel, OCR, MEI, Eduqas)

At time t = 0 hours, ship A is at the point with position vector $(10\mathbf{i} + 12\mathbf{j})$ km relative to a port O, and moves with constant velocity $(-2\mathbf{i} - 2\mathbf{j})$ km h^{-1}. Ship B leaves O at t = 0 with constant speed 5 km h^{-1} and intercepts A.
Find (a) the time at which B intercepts A, (b) the velocity of B as a vector.

Position of A at time t: $(10\mathbf{i} + 12\mathbf{j}) + t(-2\mathbf{i} - 2\mathbf{j})$. B at time t: t v_B (unknown velocity)
At interception B has travelled 5 t, so $|(10\mathbf{i} + 12\mathbf{j}) + t(-2\mathbf{i} - 2\mathbf{j})| = 5\,t$
Squaring and rearranging: $-17t^2 - 88t + 244 = 0$, which gives t = 2 or a negative value
t = 2 (the other root is negative since B is faster than A)

A is then at $(10\mathbf{i} + 12\mathbf{j}) + 2(-2\mathbf{i} - 2\mathbf{j}) = (6\mathbf{i} + 8\mathbf{j})$, so $v_B = \frac{1}{2}(6\mathbf{i} + 8\mathbf{j}) = (3\mathbf{i} + 4\mathbf{j})$

Answer: (a) 2 h (b) $(3\mathbf{i} + 4\mathbf{j})$ km h^{-1}

7 Normal then binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The size of a battery is $N(116, 9^2)$ hours. A battery is 'large' if it exceeds 125.
(a) Find the probability that a battery is large.
(b) 20 batterys are chosen at random. Find the probability that at least 2 are large.

(a) $P(X > 125) = P(Z > 1.000) = 0.1587$
(b) Let Y = number large, $Y \sim B(20, 0.1587)$
$P(Y \geq 2) = 1 - P(Y \leq 1) = 0.849$
Answer: (a) 0.159 (b) 0.849

8 Quadratic sequence from recurrence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = u_n + 4n - 2$, $u_1 = 0$.
(a) Show that $u_n = an^2 + bn + c$ and find a, b, c.
(b) Find the least n for which $u_n > 833$.

$$u_n = u_1 + \sum_{k=1}^{n-1}(4k - 2) = 0 + 4 \times \frac{(n-1)n}{2} - 2(n-1).$$

$= 2n^2 - 4n + 2$.
So a = 2, b = −4, c = 2.
$u_{21} = 800 \leq 833$ and $u_{22} = 882 > 833$.
Answer: (a) a = 2, b = −4, c = 2 (b) n = 22

9 Stationary points: exact coordinates (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact coordinates of the stationary point of
$y = x \ln x$ for $x > 0$,
and determine its nature.

$dy/dx = (\ln x + 1) = 0$ gives $\ln x = -\frac{1}{1}$

$y = -e^{-1}$ when $x = e^{-1}$
Second derivative at this point is positive, so it is a minimum
Answer: $(e^{-1}, -e^{-1})$, minimum

10 Partial fractions (improper) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{3x^3 - x^2 - 17x - 2}{(x-2)(x+2)}$ in the form $ax + b + \frac{C}{(x-2)} + \frac{D}{(x+2)}$.

Divide: the numerator has degree 3, the denominator degree 2.
$(3x^3 - x^2 - 17x - 2) \div (x^2 - 4)$ gives quotient $3x - 1$.
Remainder: $-5x - 6$.
Put x = 2: remainder = C(4) so C = −4. Put x = −2: D = −1.
Answer: $3x - 1 - \frac{4}{(x-2)} - \frac{1}{(x+2)}$