Day 313 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Resultant of forces: unknown constant and acceleration (AQA, Edexcel, OCR, MEI, Eduqas)

Three forces $(4\mathbf{i} - 2\mathbf{j})$ N, $(-2\mathbf{i} + k\mathbf{j})$ N and $(-\mathbf{i} + 6\mathbf{j})$ N act on a particle of mass 4 kg. The resultant is parallel to $(\mathbf{i} + 3\mathbf{j})$. Find (a) the value of k, (b) the magnitude of the acceleration of the particle, to 3 s.f.

Resultant = $(1)\mathbf{i} + (4 + k)\mathbf{j}$

Parallel to $(\mathbf{i} + 3\mathbf{j})$: $\frac{4 + k}{1} = \frac{3}{1}$, so $k = -1$

R = $(\mathbf{i} + 3\mathbf{j})$ N, so $|R| = \sqrt{(1^2 + 3^2)} = 3.162$

a = |R| ÷ 4

Answer: (a) k = −1 (b) 0.791 m s^{-2}

2 Integrals of the form f'(x)/f(x) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact value of

$$\int_0^3 \frac{6x + 3}{x^2 + x + 5}\,dx$$

Numerator = 3 × derivative of denominator

Integral = $[3 \ln(x^2 + x + 5)]$

= 3 (ln 17 − ln 5)

Answer: 3ln 17 − 3ln 5

3 Arithmetic series from a sum formula (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

An arithmetic sequence has $u_6 = 31$ and $S_{12} = 408$.

(a) Find the first term and common difference.

(b) Find the least n for which $S_n > 360$.

u_6: a + 5d = 31. S_{12}: 12(2a + 11d)/2 = 408, so 12a + 66d = 408.

Solving: a = 1, d = 6.

$S_n = \frac{n}{2}(2 + 6n - 6) = \frac{n(6n - 4)}{2}$.

Test: $S_{11} = 341 \leq 360$, $S_{12} = 408 > 360$.

Answer: (a) a = 1, d = 6 (b) n = 12

4 Large data set: linear change of units (AQA, Edexcel, OCR, MEI, Eduqas)

Large Data Set, daily mean air temperature (°C), Heathrow, August 2015 (n = 184): mean 19.4, sd 2.8. (F = 1.8C + 32)

(a) Find the mean and sd in °F.

(b) Is a reading of 27 more than 2 sd above the mean?

New value = 1.8 × old + 32, so mean = 1.8 × 19.4 + 32 = 66.92

The constant shifts the mean but not the spread; only the multiplier affects σ

σ = 1.8 × 2.8 = 5.040

Outlier cut-off = mean + 2σ = 19.4 + 5.6 = 25 (original units)

27 is greater than 25, so yes

Answer: (a) mean 66.9, sd 5.04 (b) yes

5 Force applied then removed (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of mass 20 kg is at rest on rough horizontal ground (μ = 0.25). A horizontal force of 77 N acts on it for 7 s and is then removed.

Find (a) the speed when the force is removed, (b) the total distance travelled before the box stops. Take g = 9.8 m s^{-2}.

Friction = μmg = 0.25 × 20 × 9.8 = 49 N

While pushed: 77 − 49 = 20 a, so a = 1.4

v = 1.4 × 7 = 9.8; s_1 = ½ × 1.4 × 7^2 = 34.3

After release: deceleration = μg = 2.45, so s_2 = $\frac{9.8^2}{2 \times 2.45}$ = 19.6

Total distance = 34.3 + 19.6

Answer: (a) 9.8 m s^{-1} (b) 53.9 m

6 Geometric series: sum to infinity (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A geometric series has sum to infinity 18 and second term -8. Find the possible values of the common ratio r and the first term a in each case.

$a = \frac{-8}{r}$. $S_\infty = \frac{a}{1-r} = \frac{-8}{r(1-r)} = 18$.

$18r^2 - 18r - 8 = 0$.

$(3r + 1)(3r - 4) = 0$: $r = -\frac{1}{3}$ or $r = \frac{4}{3}$.

Need $|r| < 1$: $-\frac{1}{3}$ valid, $\frac{4}{3}$ rejected.

Answer: $r = -\frac{1}{3}$, $a = 24$

7 R-form: greatest and least values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(\theta) = \frac{4}{26 + 24\sin\theta + 7\cos\theta}$

(a) Write $24\sin\theta + 7\cos\theta$ as $R\sin(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.

(b) Find the greatest and least values of $f(\theta)$, and the smallest positive θ at which the greatest value occurs, to 1 d.p.

$R = \sqrt{24^2 + 7^2} = 25$, $\tan\alpha = \frac{7}{24}$, $\alpha = 16.3°$

Denominator $= 26 + 25\sin(\theta + 16.3°)$, which lies between 1 and 51

Greatest $f(\theta) = \frac{4}{1} = 4$, when $\sin(\theta + \alpha) = -1$

Least $f(\theta) = \frac{4}{51} = \frac{4}{51}$, when $\sin(\theta + \alpha) = 1$

$\sin(\theta + \alpha) = -1$: $\theta + 16.3° = 270°$, so $\theta = 253.7°$

Answer: (a) $25\sin(\theta + 16.3°)$ (b) greatest 4, least $\frac{4}{51}$, $\theta = 253.7°$

8 Partial fractions then integrate (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Show that $\frac{x^2 + 11x + 23}{(x + 4)(x + 3)} \equiv 1 + \frac{A}{x + 4} + \frac{B}{x + 3}$.

(b) Hence find the exact value of $\int_0^2 \frac{x^2 + 11x + 23}{(x + 4)(x + 3)}\,dx$.

Numerator $= (x + 4)(x + 3) + A(x + 3) + B(x + 4)$
$x = -4$ gives $A = 5$; $x = -3$ gives $B = -1$
Integral $= [x + 5\ln(x + 4) - 1\ln(x + 3)]$
Substitute limits 0 and 2
Answer: $A = 5$, $B = -1$; $2 - 5\ln 2 + 6\ln 3 - \ln 5$

9 Circle through three points (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A circle passes through P(−13, 12), Q(−4, 15) and R(−16, −9). Find its equation.

Centre is on the perpendicular bisector of each chord.
Bisector of PQ: $9x + 3y = -36$.
Bisector of QR: $-12x - 24y = 48$.
Solving: centre $(-4, 0)$.
$r^2 = (-13 - (-4))^2 + (12 - 0)^2 = 225$.
Answer: $(x + 4)^2 + (y + 0)^2 = 225$

10 Conditional probability from given values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

For events A and B, P(A) = 0.3, P(B) = 0.1 and P(A | B) = 0.5.

(a) Find P(A ∩ B).

(b) Find P(A | B′).

$P(A \cap B) = P(A \mid B) \times P(B) = 0.5 \times 0.1 = 0.05$

$P(A \cup B) = 0.3 + 0.1 - 0.05 = 0.35$

$P(B') = 0.9$

$P(A \cap B') = P(A) - P(A \cap B) = 0.25$

$P(A \mid B') = \frac{0.25}{0.9}$

Answer: (a) 0.05 (b) $\frac{5}{18}$