Day 312 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Cumulative distribution function (CIE)

The cdf of X is F(x) = 0 for x < 0, $F(x) = 1 - (1 - \frac{x}{4})^3$ for $0 \le x \le 4$, and F(x) = 1 for x > 4.

(a) Find the pdf of X.
(b) Find E(X).
(c) Find P(1 < X < 2) and the median, to 3 significant figures.

(a) $f(x) = F'(x) = \frac{3}{4}(1 - \frac{x}{4})^2$ for $0 \le x \le 4$

(b) $E(X) = \int x\, f(x)\, dx$ from 0 to 4 = 1

(c) P = F(2) − F(1) = 0.2969

Median m: F(m) = 1/2, so $m = 4(1 - (1/2)^{1/3})$

m = 0.8252

Answer: (a) $f(x) = \frac{3}{4}(1 - \frac{x}{4})^2$ (b) 1 (c) 0.297, median 0.825

2 Velocity parallel to a given direction (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves in a plane. At time t seconds its position vector relative to a fixed origin O is r metres, where $\mathbf{r} = (-2.5t^2 - 7t)\mathbf{i} + (-3t^2 + 7t)\mathbf{j}$.

Find (a) the value of t at which P is moving parallel to $(2\mathbf{i} + \mathbf{j})$, (b) the speed of P at this time (3 s.f. if not exact).

$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-5t - 7)\mathbf{i} + (-6t + 7)\mathbf{j}$

Parallel to $(2\mathbf{i} + \mathbf{j})$: $(-5t - 7) \times 1 = (-6t + 7) \times 2$

t(7) = 21, so t = 3

At t = 3: $\mathbf{v} = (-22\mathbf{i} - 11\mathbf{j})$, speed $= \sqrt{(-22^2 + -11^2)}$

Answer: (a) t = 3 (b) 24.6 m s^{-1}

3 Exponential growth and decay models (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A radioactive sample has mass $m = m_0 e^{-kt}$ grams, t in days.

Initially m = 500 and after 4 days m = 300.

Find k, and the time for the mass to fall to 100 grams, to 3 s.f.

$m_0 = 500$. $500\, e^{-4k} = 300$ so $e^{-4k} = \frac{3}{5}$

$k = \frac{1}{4} \ln \frac{5}{3} = 0.128$

$500\, e^{-kt} = 100$ so $e^{kt} = 5$, $t = \frac{1}{k} \ln 5$

t = 12.6 days

Answer: k = 0.128, t = 12.6 days

4 Surds: rationalising the denominator (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{2 + 5\sqrt{5}}{1 - \sqrt{5}}$ in the form $p + q\sqrt{5}$, where p and q are rational numbers.

Multiply top and bottom by the conjugate $(1 + \sqrt{5})$.

Numerator: $(2 + 5\sqrt{5})(1 + \sqrt{5}) = 27 + 7\sqrt{5}$.

Denominator: $1^2 - 1^2(5) = -4$.

So the fraction = $\frac{27 + 7\sqrt{5}}{-4}$.

Answer: $-\frac{27}{4} - \frac{7}{4}\sqrt{5}$

Day 312 **A Level Maths: S Tier 10-a-day: Model Solutions**

5 Newton cooling model (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A drink cools according to $\theta = 25 + Ae^{-kt}$, where θ °C is the temperature after t minutes. Initially $\theta = 85$, and after 8 minutes $\theta = 55$.
Find A and k, then find when $\theta = 37$ (3 s.f.).

$t = 0$: $25 + A = 85$ so $A = 60$

$t = 8$: $25 + 60e^{-8k} = 55$ so $e^{-8k} = \frac{1}{2}$

$k = \frac{1}{8} \ln 2 = 0.0866$

$60\, e^{-kt} = 12$ so $t = \frac{1}{k} \ln 5$

$t = 18.6$ minutes

Answer: A = 60, k = 0.0866, t = 18.6 minutes

6 Parametric curve and line intersection (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has parametric equations
$x = 2t$, $y = 3t^2$. The line $y = -\frac{3}{2}x + 18$
meets C at A and B. Find the exact length AB.

Substitute: $3t^2 = -3t + 18$.
$3t^2 + 3t - 18 = 0 \Rightarrow t = -3$ or $t = 2$.
A(−6, 27), B(4, 12).
$AB^2 = (-10)^2 + 15^2 = 325$.
Answer: AB = $5\sqrt{13}$

7 Normal then binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The size of a rod is $N(87, 3^2)$ mm. A rod is 'large' if it exceeds 90.
(a) Find the probability that a rod is large.
(b) 12 rods are chosen at random. Find the probability that at least 4 are large.

(a) $P(X > 90) = P(Z > 1.000) = 0.1587$
(b) Let Y = number large, $Y \sim B(12, 0.1587)$
$P(Y \geq 4) = 1 - P(Y \leq 3) = 0.109$
Answer: (a) 0.159 (b) 0.109

8 Related rates of change (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The volume of a sphere increases at a constant rate of 50π cm^3s^{-1}.
When the radius is 5 cm, find the rate of increase of (a) the radius,
(b) the surface area. Volume = $\frac{4}{3}\pi r^3$.

$dV/dt = 4\pi r^2\, dr/dt$, so $50\pi = 4\pi \times 25 \times dr/dt$

$dr/dt = \frac{1}{2}$

$A = 4\pi r^2$, so $dA/dt = 8\pi r\, dr/dt = 8\pi \times 5 \times \frac{1}{2}$

Answer: (a) $\frac{1}{2}$ cm s^{-1} (b) 20π cm^2s^{-1}

Day 312 **A Level Maths: S Tier 10-a-day: Model Solutions**

9 Horizontal force on a rough slope: range of values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle of weight 145 N rests on a rough plane inclined at α to the horizontal, where $\tan\alpha = \frac{4}{3}$ and $\mu = 0.5$.

A horizontal force H N, in the vertical plane through a line of greatest slope, holds it in equilibrium. Find the range of possible values of H.

$\sin\alpha = \frac{4}{5}$, $\cos\alpha = \frac{3}{5}$. Normal reaction: $R = 145\cos\alpha + H\sin\alpha$

About to slide down: $H\cos\alpha + \mu R = 145\sin\alpha$, so $H(\frac{3}{5} + 0.5 \times \frac{4}{5}) = 145(\frac{4}{5} - 0.5 \times \frac{3}{5})$

$H_{min} = 72.5$

About to slide up: $H\cos\alpha = 145\sin\alpha + \mu R$, so $H(\frac{3}{5} - 0.5 \times \frac{4}{5}) = 145(\frac{4}{5} + 0.5 \times \frac{3}{5})$

$H_{max} = 797.5$

Answer: $72.5 \leq H \leq 797.5$

10 Sums of multiples (arithmetic series) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the sum of all the integers from 1 to 760
inclusive that are divisible by 4 or by 10, but not by both.

Multiples of 4: 190 terms (4 to 760): sum = 72580.
Multiples of 10: 76 terms: sum = 29260.
Multiples of both, i.e. of 20: 38 terms: sum = 14820.
Required sum = $72580 + 29260 - 2 \times 14820$.

Answer: 72200