Day 311 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Trig equations with compound angles (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\sin(3\theta - 30°) = -\frac{\sqrt{3}}{2}$
for $0° \leq \theta \leq 360°$.

Let $u = 3\theta - 30°$; then $\sin u = -\frac{\sqrt{3}}{2}$
Base solutions: $u = 240, -60°$ (+ multiples of 360°)
$\theta = (u + 30°) \div 3$, keeping those in range
Answer: θ = 90°, 110°, 210°, 230°, 330°, 350°

2 Replacing an observation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

20 recorded times (seconds) have mean 34 and standard deviation 8.
One value was recorded as 41 but should have been 59.
Find the correct mean and the correct standard deviation.

$\Sigma x = 20 \times 34 = 680$, so corrected $\Sigma x = 680 - 41 + 59 = 698$

Correct mean $= \frac{698}{20} = 34.90$

$\Sigma x^2 = 20(64 + 1156) = 24400$, corrected $\Sigma x^2 = 24400 - 1681 + 3481 = 26200$

Variance $= \frac{26200}{20} - 34.900^2 = 91.990$, $\sigma = 9.591$

Answer: mean 34.9, standard deviation 9.59

3 Equations quadratic in an exponential (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $2^{2x} - 24 \times 2^x + 128 = 0$,
giving each answer exactly and, where not an integer, to 3 s.f.

Let $u = 2^x$: $u^2 - 24u + 128 = 0$
$(u - 8)(u - 16) = 0$, so $2^x = 8$ or $2^x = 16$
Take logarithms: $x = \log_2 8 = 3$; $x = \log_2 16 = 4$
Answer: $x = \log_2 8 = 3$; $x = \log_2 16 = 4$

4 Partial fractions (repeated factor) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{-2x^2 + 24x - 67}{(x-1)(x-4)^2}$ in the form
$\frac{A}{(x-1)} + \frac{B}{(x-4)} + \frac{C}{(x-4)^2}$.

$-2x^2 + 24x - 67 \equiv A(x-4)^2 + B(x-1)(x-4) + C(x-1)$.
Put $x = 1$: $-45 = A(-3)^2$, so $A = -5$.
Put $x = 4$: $-3 = C(3)$, so $C = -1$.
Compare x^2: $-2 = 1A + 1B$, so $B = 3$.
Answer: A = −5, B = 3, C = −1

5 Coded data: mean, standard deviation, Σx^2 (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

For 20 observations x, $\Sigma(x - 50) = 90$ and $\Sigma(x - 50)^2 = 805$.
(a) Find the mean of x.
(b) Find the standard deviation of x.
(c) Find the value of Σx^2.

Mean of $(x - 50) = \frac{90}{20} = 4.5$, so mean of $x = 54.5$

Variance $= \frac{805}{20} - (4.5)^2 = 20$

$\sigma = \sqrt{20}$

$\Sigma x^2 = \Sigma(x - 50)^2 + 100\,\Sigma(x - 50) + 20 \times 50^2$
$= 805 + 9000 + 50000$
Answer: (a) 54.5 (b) 4.47 (c) 59805

6 Three-stage journey (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A train accelerates uniformly from rest at 0.75 m s^{-2} to a speed of V m s^{-1}, travels at this speed, then decelerates uniformly at 1.25 m s^{-2} to rest.
The journey takes 73 s and covers 855 m. Find V and the time spent at constant speed.

Time accelerating = $\frac{V}{0.75}$, time decelerating = $\frac{V}{1.25}$, so time at constant speed = $73 - \frac{V}{0.75} - \frac{V}{1.25}$

Area under the speed-time graph: $855 = 73V - \frac{V^2}{1.5} - \frac{V^2}{2.5}$

Multiply by 2ab: $2V^2 - 136.875V + 1603.125 = 0$
$V = 15$ or $V = 53.4375$
$V = 53.4375$ gives time at constant speed -41 s, which is negative, so $V = 15$
Time at constant speed = $73 - 20 - 12 = 41$ s
Answer: V = 15, 41 s

7 Inverse function and range (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The function f is defined by $f(x) = 2x^2 - 4x - 2$
for $x \geq k$. Given that k is the least value for which
f is one-one, find k, $f^{-1}(x)$, and the domain of f^{-1}.

Complete the square: $f(x) = 2(x - 1)^2 - 4$.
The vertex is at $x = 1$, so $k = 1$ (f is one-one on one side of it).

Let $y = f(x)$: $y + 4 = 2(x - 1)^2$, so $(x - 1) = \pm\sqrt{\frac{y + 4}{2}}$.

As $x \geq 1$, take the positive root: $x = 1 + \sqrt{\frac{y + 4}{2}}$.

Range of f = domain of f^{-1}: $f(1) = -4$, so $x \geq -4$.

Answer: $k = 1$; $f^{-1}(x) = 1 + \sqrt{\frac{x + 4}{2}}$; $x \geq -4$

8 Boat crossing a river (AQA, Edexcel, OCR, MEI)

A river is 120 m wide and flows at 6 m s^{-1}. A boatman can row at 10 m s^{-1} in still water and wants to reach the point directly opposite his starting point.
Find (a) the angle to the bank at which he should head, to 1 d.p., (b) the time taken, to 3 s.f.

Resultant velocity must be perpendicular to the bank, so the upstream component of the boat's velocity cancels the current
Let the boat head at angle φ upstream from the perpendicular: $10 \sin \varphi = 6$, so $\varphi = 36.9°$
Angle to the bank = $90° - \varphi = 53.1°$
Speed across = $\sqrt{(10^2 - 6^2)} = 8$ m s^{-1}
Time = $\frac{120}{8} = 15.0$ s
Answer: (a) 53.1° (b) 15.0 s

9 Stationary points: exact coordinates (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact coordinates of the stationary point of
$y = 4x^3 \ln x$ for $x > 0$,
and determine its nature.

$dy/dx = 4x^2(3 \ln x + 1) = 0$ gives $\ln x = -\frac{1}{3}$

$y = -\frac{4}{3}e^{-1}$ when $x = \frac{1}{\sqrt[3]{e}}$

Second derivative at this point is positive, so it is a minimum

Answer: $(\frac{1}{\sqrt[3]{e}}, -\frac{4}{3}e^{-1})$, minimum

10 Equidistant point and triangle area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A(−15, 5) and B(−3, −13). The point C lies on the x-axis and CA = CB. Find the coordinates of C and the area of triangle ABC.

C = (c, 0): $(c - (-15))^2 + 25 = (c - (-3))^2 + 169$.
Expand and solve: $24c = -72$, so $c = -3$.
Area = $\frac{1}{2}|(x_A - c)y_B - (x_B - c)y_A|$
$= \frac{1}{2}|(-12) \times (-13) - 0 \times 5| = 78$.
Answer: C(−3, 0); area 78