Day 310 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Trig equations with compound angles (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\cot(2\theta - 30°) = \frac{1}{\sqrt{3}}$ for $0° \le \theta \le 180°$.

Let $u = 2\theta - 30°$; then $\cot u = \frac{1}{\sqrt{3}}$

Base solutions: $u = 60°$ (+ multiples of 180°)

$\theta = (u + 30°) \div 2$, keeping those in range

Answer: $\theta = 45°, 135°$

2 Regression line from summations (AQA, Edexcel, OCR, MEI)

For n = 8 pairs of values: $\Sigma x = 120$, $\Sigma y = 264$, $\Sigma x^2 = 1962$, $\Sigma xy = 3636$.

(a) Find the equation of the regression line of y on x.

(b) The x values lie between 11 and 20. Estimate y when x = 24 and comment on its reliability.

$S_{xx} = 1962 - \frac{120^2}{8} = 162$

$S_{xy} = 3636 - \frac{120 \times 264}{8} = -324$

$b = \frac{S_{xy}}{S_{xx}} = \frac{-324}{162} = -2$

$a = \bar{y} - b\,\bar{x} = 33 - (-2)(15) = 63$

$y = 63 - 2x$

At x = 24: y = 15. This is extrapolation (outside the data range), so may be unreliable

Answer: (a) $y = 63 - 2x$ (b) 15, unreliable (extrapolation)

3 Surds: rationalising the denominator (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{2 + 2\sqrt{2}}{2 - 2\sqrt{2}}$ in the form $p + q\sqrt{2}$, where p and q are rational numbers.

Multiply top and bottom by the conjugate $(2 + 2\sqrt{2})$.

Numerator: $(2 + 2\sqrt{2})(2 + 2\sqrt{2}) = 12 + 8\sqrt{2}$.

Denominator: $2^2 - 2^2(2) = -4$.

So the fraction = $\frac{12 + 8\sqrt{2}}{-4}$.

Answer: $-3 - 2\sqrt{2}$

4 Combined mean and standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Class S has 30 students with mean mark 65 and standard deviation 3. Class P has 40 students with mean mark 71 and standard deviation 9.

(a) Find the mean mark of all 70 students.

(b) Find the standard deviation of all 70 students.

Σx for S = $30 \times 65 = 1950$, Σx for P = $40 \times 71 = 2840$

Combined mean = $\frac{4790}{70} = 68.4$

$\Sigma x^2 = n(\sigma^2 + \text{mean}^2)$: $30(9 + 4225) + 40(81 + 5041) = 331900$

Variance = $\frac{331900}{70} - 68.429^2 = 58.959$, so $\sigma = 7.678$

Answer: (a) 68.4 (b) 7.68

5 Newton cooling model (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A drink cools according to $\theta = 25 + Ae^{-kt}$, where θ °C is the temperature after t minutes. Initially $\theta = 85$, and after 10 minutes $\theta = 55$.
Find A and k, then find when $\theta = 31$ (3 s.f.).

$t = 0$: $25 + A = 85$ so $A = 60$

$t = 10$: $25 + 60e^{-10k} = 55$ so $e^{-10k} = \frac{1}{2}$

$k = \frac{1}{10} \ln 2 = 0.0693$

$60\, e^{-kt} = 6$ so $t = \frac{1}{k} \ln 10$

$t = 33.2$ minutes

Answer: A = 60, k = 0.0693, t = 33.2 minutes

6 Equations with a square root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\sqrt{4x + 4} = x - 7$.

Square both sides: $4x + 4 = (x - 7)^2$.

$x^2 - 18x + 45 = 0$, so $x = 3$ or $x = 15$.

Check in $\sqrt{4x + 4} = x - 7$: $x = 15$ gives $8 \geq 0$ (valid).

$x = 3$ gives right-hand side $-4 < 0$, so it is rejected.

Answer: x = 15

7 Non-uniform rod: centre of mass and tilting (AQA, Edexcel, OCR, MEI)

A non-uniform rod AB of length 8 m and mass 5 kg rests on supports at C and D, where AC = 2 m and AD = 6 m, with $R_C = 1.5 R_D$.
Find (a) the distance of the centre of mass from A. A particle of mass m kg at B puts the rod on the point of tilting about D. (b) Find m.

$R_C = 1.5 R_D$. Moments about A: $R_C \times 2 + R_D \times 6 = 5g \times AG$

$R_D(1.5 \times 2 + 6) = 5g \times AG$ with $R_D(1 + 1.5) = 5g$, so $AG = \frac{1.5 \times 2 + 6}{2.5} = 3.6$

Tilting about D: $R_C = 0$. Moments about D: $mg(8 - 6) = 5g(6 - 3.6)$

$m = \frac{5 \times 2.4}{2} = 6$

Answer: (a) 3.6 m (b) 6

8 Triangle in 3D: angle and area (CIE (9709); Further Maths on other boards)

The points A, B, C have coordinates $(-2, 4, 1)$, $(2, 3, -3)$, $(-3, 1, 2)$.
Find angle ABC to 1 d.p., and the exact area of triangle ABC.

$\mathrm{BA} = \begin{pmatrix} -4 \\ 1 \\ 4 \end{pmatrix}$, $\mathrm{BC} = \begin{pmatrix} -5 \\ -2 \\ 5 \end{pmatrix}$; $\mathrm{BA} \cdot \mathrm{BC} = 38$

$|\mathrm{BA}| = \sqrt{33}$, $|\mathrm{BC}| = \sqrt{54}$

$\cos \mathrm{ABC} = \frac{38}{\sqrt{33}\sqrt{54}} = 0.9002$, so $\mathrm{ABC} = 25.8°$

$|\mathrm{BA} \times \mathrm{BC}| = \left|\begin{pmatrix} 13 \\ 0 \\ 13 \end{pmatrix}\right| = \sqrt{338}$

Area $= \frac{1}{2}|\mathrm{BA} \times \mathrm{BC}|$

Answer: ABC = 25.8°, area = $\frac{13}{2}\sqrt{2}$

9 Tangent line to circle: finding k (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The line $5y = 12x + k$ is a tangent to the circle $(x - 1)^2 + (y - 1)^2 = 1$.
Find the two possible values of k.

Substitute $y = \frac{12x + k}{5}$: $(x - 1)^2 + (\frac{12x + k}{5} - 1)^2 = 1$.

Multiply by 25 and collect: a quadratic in x with a = 169.

Tangent ⇒ discriminant = 0. This simplifies to $(k + 7)^2 = 169$.

$k + 7 = \pm 13$.

Answer: k = −20 or k = 6

10 Block on a plank: limiting slip (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A plank of mass 8 kg lies on a smooth horizontal table, with a block of mass 2 kg on it ($\mu = 0.4$ between them). A horizontal force P N acts on the block.
(a) Find the greatest P for which the block does not slip. (b) Find the accelerations of block and plank when P = 29.4.
Take $g = 9.8$ m s^{-2}.

No slipping: common acceleration $a = \frac{P}{10}$. The plank is accelerated only by friction: $F = 8a$

Limiting friction: $F = \mu mg = 0.4 \times 2 \times 9.8 = 7.84$, so $a = \frac{7.84}{8} = 0.98$

P_max = 10 × 0.98 = 9.8

P = 29.4 > 9.8, so the block slips. Plank: $a = \frac{7.84}{8} = 0.98$

Block: $29.4 - 7.84 = 2a$, so $a = 10.78$

Answer: (a) 9.8 N (b) block 10.78 m s^{-2}, plank 0.98 m s^{-2}