Day 31 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Large data set: linear change of units (AQA, Edexcel, OCR, MEI, Eduqas)

Large Data Set, daily mean air temperature (°C), Hurn, June 2015 (n = 31): mean 12.6, sd 4. (F = 1.8C + 32)
(a) Find the mean and sd in °F.
(b) Is a reading of 17.4 more than 2 sd above the mean?

New value = 1.8 × old + 32, so mean = 1.8 × 12.6 + 32 = 54.68
The constant shifts the mean but not the spread; only the multiplier affects σ
σ = 1.8 × 4 = 7.200
Outlier cut-off = mean + 2σ = 12.6 + 8 = 20.6 (original units)
17.4 is less than 20.6, so no
Answer: (a) mean 54.7, sd 7.20 (b) no

2 Parametric curve and line intersection (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has parametric equations
$x = t$, $y = 4t^2$. The line $y = 8x + 12$
meets C at A and B. Find the exact length AB.

Substitute: $4t^2 = 8t + 12$.
$4t^2 - 8t - 12 = 0 \Rightarrow t = 3$ or $t = -1$.
A(3, 36), B(−1, 4).
$AB^2 = 4^2 + 32^2 = 1040$.
Answer: AB = $4\sqrt{65}$

3 Normal: unknown standard deviation then a second probability (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The scores in an aptitude test are modelled by $N(78, \sigma^2)$. The probability of a value below 62 is 0.10.
(a) Find σ.
(b) Find the probability of a value above 69.

$P(X < 62) = 0.10$ so $Z = \frac{62 - 78}{\sigma} = -1.2816$

$\sigma = \frac{-16}{-1.2816} = 12.48$

$z = \frac{69 - 78}{12.48} = -0.721$

$P(X > 69) = P(Z > -0.721) = 0.765$
Answer: (a) 12.5 (b) 0.765

4 Velocity-time graph with a return (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the velocity-time graph of a particle P moving on a straight line (velocity in m s^{-1}, time in s). At time 17 s, P is back at its starting point.
Find (a) the value of W, (b) the greatest distance of P from its starting point.

Net area = 0 (back at start): ½ × 4 × 6 + ½(6 − W)(6) − W(7) = 0
12 + 18 − 3W − 7W = 0, so 10W = 30
W = 3

v = 0 where the line from (4, 6) to (10, −3) crosses the axis: $t = 4 + 6 \times \frac{6}{9} = 8$

Greatest distance = area of triangle = ½ × 8 × 6
Answer: (a) W = 3 (b) 24 m

5 Triple angle equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE (triple angle))

Given that $\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$, solve
$\sin 3\theta = 0$ for $0 < \theta < 180°$.

Substitute: $\sin\theta(3 - 4\sin^2\theta - 0) = 0$

$\sin\theta = 0$ or $\sin^2\theta = \frac{3}{4}$

$\sin\theta = 0$ gives no solutions in the range

$\sin^2\theta = \frac{3}{4}$ gives $60°$, $120°$

Answer: $\theta = 60°, 120°$

6 Direction of motion at 45 degrees (AQA, Edexcel, OCR, MEI, Eduqas)

A shell is projected from level ground with horizontal and vertical velocity components 49 and 63.7 m s^{-1}.
Find (a) the two times at which it moves at 45° to the horizontal, (b) the horizontal distance between those two points.
Take g = 9.8 m s^{-2}.

At 45° to the horizontal, $|v_y| = v_x = 49$
Moving upwards: $63.7 - 9.8t = 49$, so $t = 1.5$
Moving downwards: $63.7 - 9.8t = -49$, so $t = 11.5$
Horizontal distance $= 49 \times (11.5 - 1.5)$
Answer: (a) 1.5 s and 11.5 s (b) 490 m

7 Quadratic sequence from recurrence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = u_n + 3n - 1$, $u_1 = -4$.
(a) Show that $u_n = an^2 + bn + c$ and find a, b, c.
(b) Find the least n for which $u_n > 256$.

$$u_n = u_1 + \sum_{k=1}^{n-1}(3k - 1) = -4 + 3 \times \frac{(n-1)n}{2} - 1(n-1).$$

$$= \frac{3}{2}n^2 - \frac{5}{2}n - 3.$$

So $a = \frac{3}{2}$, $b = -\frac{5}{2}$, $c = -3$.

$u_{14} = 256 \leq 256$ and $u_{15} = 297 > 256$.

Answer: (a) $a = \frac{3}{2}$, $b = -\frac{5}{2}$, $c = -3$ (b) $n = 15$

8 R-form: greatest and least values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(\theta) = \frac{2}{10 + 4\sin\theta + 3\cos\theta}$
(a) Write $4\sin\theta + 3\cos\theta$ as $R\sin(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.
(b) Find the greatest and least values of $f(\theta)$, and the smallest positive θ at which the greatest value occurs, to 1 d.p.

$R = \sqrt{4^2 + 3^2} = 5$, $\tan\alpha = \frac{3}{4}$, $\alpha = 36.9°$

Denominator $= 10 + 5\sin(\theta + 36.9°)$, which lies between 5 and 15

Greatest $f(\theta) = \frac{2}{5} = \frac{2}{5}$, when $\sin(\theta + \alpha) = -1$

Least $f(\theta) = \frac{2}{15} = \frac{2}{15}$, when $\sin(\theta + \alpha) = 1$

$\sin(\theta + \alpha) = -1$: $\theta + 36.9° = 270°$, so $\theta = 233.1°$

Answer: (a) $5\sin(\theta + 36.9°)$ (b) greatest $\frac{2}{5}$, least $\frac{2}{15}$, $\theta = 233.1°$

9 Binomial expansion (fractional or negative powers) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the binomial expansion of $(8 - 6x)^{-\frac{1}{3}}$ in ascending powers of x up to and including the term in x^3.
State the values of x for which it is valid.

$(8 - 6x)^{-\frac{1}{3}} = 8^{-\frac{1}{3}}(1 - \frac{3}{4}x)^{-\frac{1}{3}} = \frac{1}{2}(1 + u)^{-\frac{1}{3}}$, $u = -\frac{3}{4}x$.

$(1 + u)^{-\frac{1}{3}} = 1 + (-\frac{1}{3})u + \frac{2}{9}u^2 + (-\frac{14}{81})u^3 + ...$

Substitute u and multiply by $\frac{1}{2}$.

Valid when $|u| < 1$, i.e. $|x| < \frac{4}{3}$.

Answer: $\frac{1}{2} + \frac{1}{8}x + \frac{1}{16}x^2 + \frac{7}{192}x^3 + ...$; $|x| < \frac{4}{3}$

10 Lines in 3D: intersection and angle (CIE (9709); Further Maths on other boards)

Line l_1: $\mathbf{r} = \begin{pmatrix}10\\1\\3\end{pmatrix} + s\begin{pmatrix}3\\2\\3\end{pmatrix}$,

line l_2: $\mathbf{r} = \begin{pmatrix}-2\\-9\\-7\end{pmatrix} + t\begin{pmatrix}-3\\-3\\-2\end{pmatrix}$.

Show whether the lines meet, giving the point of intersection if they do, and find the acute angle between them to 1 d.p.

Equate components: $10 + 3s = -2 - 3t$; $1 + 2s = -9 - 3t$; $3 + 3s = -7 - 2t$
Solving the first two: $s = -2$, $t = -2$; the third equation is satisfied
Intersection: $(4, -3, -3)$

$\cos\varphi = \frac{|-21|}{\sqrt{22}\sqrt{22}} = 0.9545$, $\varphi = 17.3°$

Answer: Lines meet at $(4, -3, -3)$; angle 17.3°