Day 309 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Trig equations with compound angles (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\text{cosec}(2\theta - 15°) = \frac{2}{\sqrt{3}}$

for $0° \leq \theta \leq 360°$.

Let $u = 2\theta - 15°$; then $\text{cosec } u = \frac{2}{\sqrt{3}}$

Base solutions: $u = 120°, 60°$ (+ multiples of 360°)

$\theta = (u + 15°) \div 2$, keeping those in range

Answer: θ = 37.5°, 67.5°, 217.5°, 247.5°

2 Partial fractions (repeated factor) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{12x^2 - 27x + 23}{(x - 3)(3x - 4)^2}$ in the form

$\frac{A}{(x - 3)} + \frac{B}{(3x - 4)} + \frac{C}{(3x - 4)^2}$.

$12x^2 - 27x + 23 \equiv A(3x - 4)^2 + B(x - 3)(3x - 4) + C(x - 3)$.

Put $x = 3$: $50 = A(5)^2$, so $A = 2$.

Put $x = \frac{4}{3}$: $\frac{25}{3} = C(-\frac{5}{3})$, so $C = -5$.

Compare x^2: $12 = 9A + 3B$, so $B = -2$.

Answer: A = 2, B = −2, C = −5

3 Without replacement: find the number of counters (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A bag has 4 red counters and some white counters, n in total.

Two are taken without replacement. P(at least one red) = $\frac{34}{55}$. Find n.

$\text{P(neither red)} = \frac{(n - 4)}{n} \times \frac{(n - 5)}{(n - 1)} = \frac{21}{55}$

$\frac{21}{55} n(n - 1) = (n - 4)(n - 5)$

Rearranging gives a quadratic in n whose roots are n = 11 and one other value

The other root is $\frac{50}{17}$ so is rejected (not a valid number of counters)

$n = 11$

Answer: n = 11

4 Newton cooling model (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A drink cools according to $\theta = 25 + Ae^{-kt}$, where θ °C is the temperature after t minutes. Initially $\theta = 80$, and after 10 minutes $\theta = 58$.
Find A and k, then find when $\theta = 54$ (3 s.f.).

$t = 0$: $25 + A = 80$ so $A = 55$

$t = 10$: $25 + 55e^{-10k} = 58$ so $e^{-10k} = \frac{3}{5}$

$k = \frac{1}{10} \ln \frac{5}{3} = 0.0511$

$55\, e^{-kt} = 29$ so $t = \frac{1}{k} \ln \frac{55}{29}$

$t = 12.5$ minutes

Answer: A = 55, k = 0.0511, t = 12.5 minutes

5 Normal approximation to a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X ~ B(200, 0.45).

(a) State why a normal approximation is suitable for X.

(b) Use a normal approximation to estimate P(X ≥ 112), to 3 significant figures.

np = 90 and nq = 110, both greater than 5 (n large, p not extreme)

Y ~ N(90, 49.5) with σ = 7.036

Apply the continuity correction:

P(X ≥ 112) ≈ P(Y > 111.5) = P(Z > 3.056) = 0.00112

Answer: 0.00112

6 Segments: exact area and perimeter (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A chord AB of a circle, centre O and radius 4 cm, subtends an angle of $\frac{2\pi}{3}$ at O.

Find the exact area and the exact perimeter of the minor segment cut off by AB.

Sector OAB: $\frac{1}{2}r^2\theta = \frac{1}{2} \times 4^2 \times \frac{2\pi}{3}$; triangle OAB: $\frac{1}{2}r^2\sin\theta = 4\sqrt{3}$

Area = $\frac{16}{3}\pi - 4\sqrt{3}$

Chord AB = $2r\sin(\theta/2) = 4\sqrt{3}$

Arc AB = $r\theta = \frac{8}{3}\pi$

Perimeter = arc + chord

Answer: Area = $\frac{16}{3}\pi - 4\sqrt{3}$ cm², perimeter = $\frac{8}{3}\pi + 4\sqrt{3}$ cm

7 Variable force in vector form (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P of mass 4 kg moves under a single force F N, where $\mathbf{F} = (24t + 12)\mathbf{i} + (12t^2 + 16)\mathbf{j}$ and t is the time in seconds.

At t = 0, P is at the origin with velocity $(3\mathbf{i} - 4\mathbf{j})$ m s^{-1}.

Find (a) the velocity of P at t = 2, (b) the position vector of P at t = 2.

$\mathbf{a} = \frac{\mathbf{F}}{4} = (6t + 3)\mathbf{i} + (3t^2 + 4)\mathbf{j}$

$\mathbf{v} = \int \mathbf{a}\,dt = (3t^2 + 3t + 3)\mathbf{i} + (t^3 + 4t - 4)\mathbf{j}$ (using $\mathbf{v}(0) = (3\mathbf{i} - 4\mathbf{j})$)

At t = 2: $\mathbf{v} = (21\mathbf{i} + 12\mathbf{j})$

$\mathbf{r} = \int \mathbf{v}\,dt = (t^3 + 1.5t^2 + 3t)\mathbf{i} + (0.25t^4 + 2t^2 - 4t)\mathbf{j}$

At t = 2: $\mathbf{r} = (20\mathbf{i} + 4\mathbf{j})$

Answer: (a) $(21\mathbf{i} + 12\mathbf{j})$ m s^{-1} (b) $(20\mathbf{i} + 4\mathbf{j})$ m

8 Recurrence: limit and convergence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = \frac{1}{4}u_n + \frac{3}{2}$, $u_1 = -4$.

(a) Find the limit L of the sequence.

(b) Find the least n such that $|u_n - L| < 0.01$.

(a) At the limit $L = \frac{1}{4}L + \frac{3}{2}$, so $L(1 - \frac{1}{4}) = \frac{3}{2}$.

L = 2.

(b) $u_n - L = (u_1 - L)(\frac{1}{4})^{n-1} = (-6)(\frac{1}{4})^{n-1}$.

Need $6 \times (\frac{1}{4})^{n-1} < 0.01$, so $n - 1 > 4.614$.

The least n is 6.

Answer: (a) L = 2 (b) n = 6

9 Interception at constant speed (AQA, Edexcel, OCR, MEI, Eduqas)

At time t = 0 hours, ship A is at the point with position vector $(32\mathbf{i} + 48\mathbf{j})$ km relative to a port O, and moves with constant velocity $(-2\mathbf{i} - 4\mathbf{j})$ km h^{-1}. Ship B leaves O at t = 0 with constant speed 10 km h^{-1} and intercepts A.
Find (a) the time at which B intercepts A, (b) the velocity of B as a vector.

Position of A at time t: $(32\mathbf{i} + 48\mathbf{j}) + t(-2\mathbf{i} - 4\mathbf{j})$. B at time t: t v_B (unknown velocity)
At interception B has travelled 10 t, so $|(32\mathbf{i} + 48\mathbf{j}) + t(-2\mathbf{i} - 4\mathbf{j})| = 10\,t$
Squaring and rearranging: $-80t^2 - 512t + 3328 = 0$, which gives t = 4 or a negative value
t = 4 (the other root is negative since B is faster than A)

A is then at $(32\mathbf{i} + 48\mathbf{j}) + 4(-2\mathbf{i} - 4\mathbf{j}) = (24\mathbf{i} + 32\mathbf{j})$, so $\mathbf{v}_B = \frac{1}{4}(24\mathbf{i} + 32\mathbf{j}) = (6\mathbf{i} + 8\mathbf{j})$

Answer: (a) 4 h (b) $(6\mathbf{i} + 8\mathbf{j})$ km h^{-1}

10 Simultaneous equations (linear and quadratic) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously:
$x + y = 1$
$2x^2 - xy + 2y^2 = 12$

From the first equation $x = 1 - y$.
Substitute into the second equation and collect terms:
this gives $5y^2 - 5y - 10 = 0$.
$y = -1$ or $y = 2$, giving $x = 2$ or $x = -1$.
Answer: (x, y) = (2, −1) and (−1, 2)