Day 306 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Binomial: mean and variance given (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A binomial random variable X has mean 14 and variance $\frac{28}{3}$.

(a) Find n and p.
(b) Find P(X = 15), correct to 3 significant figures.

$np = 14$ and $np(1 - p) = \frac{28}{3}$

Divide: $1 - p = \frac{28}{3} \div 14 = \frac{2}{3}$, so $p = \frac{1}{3}$

$n = 14 \div \frac{1}{3} = 42$

$P(X = 15) = C(42, 15) (\frac{1}{3})^{15} (\frac{2}{3})^{27} = 0.121$

Answer: (a) n = 42, p = $\frac{1}{3}$ (b) 0.121

2 Cubic with only one real root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 + 4x^2 + 13x + 18$.
Show that (x + 2) is a factor of f(x), and that the equation f(x) = 0 has exactly one real root.

$f(-2) = -8 + 16 - 26 + 18 = 0$, so (x + 2) is a factor.
$f(x) = (x + 2)(x^2 + 2x + 9)$.
For $x^2 + 2x + 9 = 0$: $b^2 - 4ac = 4 - 36 = -32 < 0$.
No real roots from the quadratic, so the only real root is $x = -2$.
Answer: Proved; the only real root is x = −2

3 Hypothesis test for a normal mean (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The journey time is N(μ, 16) with known variance 16. The mean was claimed to be 83 minutes.
A random sample of 64 has mean 82.5. Test at the 1% level whether the mean is greater than 83.

H_0: μ = 83, H_1: μ > 83. Under H_0, the sample mean ~ N(83, $\frac{16}{64}$)

$z = \frac{82.5 - 83}{4 \div \sqrt{64}} = -1.000$
p-value = P(Z > −1.000) = 0.841
0.841 > 1%, so do not reject H_0
There is insufficient evidence at the 1% level that the mean has changed as claimed
Answer: Do not reject H_0 (z = −1.000, p-value 0.841)

4 Lift, passenger and cable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A lift of mass 450 kg, carrying a passenger of mass 65 kg, is supported by a vertical cable. The lift is moving downwards and accelerating at 2.4 m s^{-2}.
Find (a) the force the floor exerts on the passenger, (b) the tension in the cable. Take g = 9.8 m s^{-2}.

Take the direction of the acceleration as positive: it is downwards
Passenger: $65 \times 9.8 - R = 65 \times 2.4$
$R = 65 \times (9.8 - 2.4) = 481$
Lift and passenger: $515 \times 9.8 - T = (450 + 65) \times 2.4$
$T = 515 \times 7.4 = 3811$
Answer: (a) 481 N (b) 3811 N

5 Sectors: perimeter and area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sector of a circle has radius r cm and angle θ radians, where $\theta < 2\pi$.
Its perimeter is 50 cm and its area is 154 cm^2.
Find the two possible pairs of values of r and θ.

Arc = rθ, so $2r + r\theta = 50$ and $\frac{1}{2}r^2\theta = 154$

$r\theta = 50 - 2r$, so $\frac{1}{2}r(50 - 2r) = 154$

$r^2 - 25r + 154 = 0$, so $(r - 11)(r - 14) = 0$

r = 11: rθ = 28, $\theta = \frac{28}{11}$ r = 14: rθ = 22, $\theta = \frac{11}{7}$

Answer: r = 11, $\theta = \frac{28}{11}$ or r = 14, $\theta = \frac{11}{7}$

6 Remainder theorem (quadratic divisor) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

When a polynomial f(x) is divided by (x + 2)
the remainder is −7; when divided by (x − 2)
the remainder is 1. Find the remainder when
f(x) is divided by (x + 2)(x − 2).

f(x) = (x + 2)(x − 2)Q(x) + px + q.
f(−2) = −7: −2p + q = −7.
f(2) = 1: 2p + q = 1.
Subtracting: −4p = −8, so p = 2, q = −3.
Answer: 2x − 3

7 Range and greatest height to speed and angle (AQA, Edexcel, OCR, MEI, Eduqas)

A golf ball is hit from a point O on horizontal ground. It lands 50 m from O, and reaches a maximum height of 25 m above the ground.
Find the angle of projection and the speed of projection, to 3 s.f. Take g = 9.8 m s^{-2}.

Vertical: $0 = u_y^2 - 2 \times 9.8 \times 25$, so $u_y = 22.14$

Time of flight = $\frac{2u_y}{9.8}$ = 4.518 s, so $u_x = \frac{50}{4.518} = 11.07$

$\tan\theta = \frac{u_y}{u_x} = 2.000$, so θ = 63.4°

Speed = $\sqrt{(u_x^2 + u_y^2)} = 24.7$

Answer: θ = 63.4°, speed 24.7 m s^{-1}

8 R-form: greatest and least values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(\theta) = \frac{3}{9 + 3\sin\theta + 4\cos\theta}$

(a) Write $3\sin\theta + 4\cos\theta$ as $R\sin(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.
(b) Find the greatest and least values of f(θ), and the smallest positive θ at which the greatest value occurs, to 1 d.p.

$R = \sqrt{3^2 + 4^2} = 5$, $\tan\alpha = \frac{4}{3}$, α = 53.1°

Denominator = $9 + 5\sin(\theta + 53.1°)$, which lies between 4 and 14

Greatest $f(\theta) = \frac{3}{4} = \frac{3}{4}$, when $\sin(\theta + \alpha) = -1$

Least $f(\theta) = \frac{3}{14} = \frac{3}{14}$, when $\sin(\theta + \alpha) = 1$

$\sin(\theta + \alpha) = -1$: $\theta + 53.1° = 270°$, so θ = 216.9°

Answer: (a) $5\sin(\theta + 53.1°)$ (b) greatest $\frac{3}{4}$, least $\frac{3}{14}$, θ = 216.9°

9 Polynomial inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $2x^3 - 10x^2 - 4x + 48 \geq 0$.

Use the factor theorem to find roots: -2, 3, 4.
$f(x) = 2(x + 2)(x - 3)(x - 4)$.
Sign of f in each region (left to right): $-$, +, $-$, +.
Take the regions where $f \geq 0$ (include roots if $\leq$ or $\geq$).
Answer: $-2 \leq x \leq 3$ or $x \geq 4$

10 Perpendicular from a point to a line (3D) (CIE (9709); Further Maths on other boards)

The line l has equation $\mathbf{r} = \begin{pmatrix} -2 \\ 9 \\ 10 \end{pmatrix} + t\begin{pmatrix} -2 \\ 3 \\ 3 \end{pmatrix}$.

The point P has coordinates $(5, -1, 10)$.

Find the coordinates of the foot F of the perpendicular from P to l, and the exact distance PF.

F = (a + td) for some t, so $\mathrm{FP} = \begin{pmatrix} 7 \\ -10 \\ 0 \end{pmatrix} - t\begin{pmatrix} -2 \\ 3 \\ 3 \end{pmatrix}$

$\mathrm{FP} \cdot \mathbf{d} = 0$: $-44 - t(22) = 0$, so $t = -2$

F = (2, 3, 4)

$\mathrm{PF} = \begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix}$, so $\mathrm{PF} = \sqrt{61}$

Answer: F = (2, 3, 4), PF = $\sqrt{61}$