Day 305 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Lift, passenger and cable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A lift of mass M kg, carrying a passenger of mass 90 kg, hangs from a vertical cable. The tension in the cable is 4840 N and the floor exerts a force of 990 N on the passenger.

Find (a) the acceleration of the lift, with its direction, (b) the value of M. Take g = 9.8 m s^{-2}.

Passenger (upwards positive): $990 - 90 \times 9.8 = 90a$, so $a = 1.2$

$a = 1.2$ m s^{-2}, so the acceleration is upwards

Lift and passenger: $4840 - (M + 90) \times 9.8 = (M + 90) \times 1.2$

$(M + 90)(9.8 + 1.2) = 4840$, so $M + 90 = \frac{4840}{11} = 440$

Answer: (a) 1.2 m s^{-2} upwards (b) M = 350

2 Transformations: new equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = 2x^2 + 5x + 2$ is translated by $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$

and then a stretch, scale factor 2, parallel to the x-axis.

Find the equation of the resulting curve in the form $y = ax^2 + bx + c$.

Translation by $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$: $y = f(x - 3) + 3 = 2x^2 - 7x + 8$.

Then a stretch, scale factor 2, parallel to the x-axis: replace x by $\frac{1}{2}x$: $y = f(\frac{1}{2}x - 3) + 3$.

Expand and simplify to give the quadratic.

Answer: $y = \frac{1}{2}x^2 - \frac{7}{2}x + 8$

3 Stratified sample then probability (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A school has 60 students in Year 10, 120 in Year 11 and 120 in the sixth form.

A stratified sample of 40 students is taken.

(a) Find the number sampled from each group.

(b) Two students are chosen at random from the sample. Find the probability they are from the same group.

Sample sizes: $40 \times \frac{60}{300}$, $40 \times \frac{120}{300}$, $40 \times \frac{120}{300} = 8, 16, 16$

P(same) = 8 × 7 + 16 × 15 + 16 × 15/40 × 39

$= \frac{536}{1560} = 0.344$

Answer: (a) 8, 16, 16 (b) 0.344

4 Range and domain by completing the square (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{10}{x^2 + 4x + 8}$ for all real x.

Find the greatest value of f(x), the value of x at which it occurs, and the range of f.

$x^2 + 4x + 8 = (x + 2)^2 + 4$.

$(x + 2)^2 \geq 0$, so the denominator is at least 4, with equality at $x = -2$.

The denominator is always positive, so $f(x) > 0$.

Greatest $f = \frac{10}{4} = \frac{5}{2}$ at $x = -2$.

Answer: greatest value $\frac{5}{2}$ at $x = -2$; range $0 < f(x) \leq \frac{5}{2}$

5 Binomial: finding p from a ratio (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X ~ B(10, p). It is given that $P(X = 2) = \frac{9}{10} \times P(X = 1)$.

(a) Find the value of p.
(b) Find P(X = 0), correct to 3 significant figures.

$\frac{90}{2} p^2(1 - p)^8 = \frac{9}{10} \times 10\, p(1 - p)^9$

Divide by $p(1 - p)^8$: $4.5\, p = \frac{9}{10}(1 - p)$

$(4.5 + \frac{9}{10})p = \frac{9}{10}$

$p = \frac{1}{6}$

$P(X = 0) = (1 - p)^{10} = (\frac{5}{6})^{10} = 0.162$

Answer: (a) $p = \frac{1}{6}$ (b) 0.162

6 Derivative of the inverse function (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve is given by $x = y^3 + 2y^2 - 2y$.
Find dy/dx and d^2y/dx^2 when $y = -1$.

$dx/dy = 3y^2 + 4y - 2 = -3$ when $y = -1$, so $dy/dx = \frac{1}{dx/dy} = -\frac{1}{3}$

$d^2x/dy^2 = -2$ when $y = -1$

$d^2y/dx^2 = -\frac{d^2x}{dy^2/(dx/dy)^3} = -\frac{-2}{(-3)^3}$

Answer: $dy/dx = -\frac{1}{3}$, $d^2y/dx^2 = -\frac{2}{27}$

7 Circle through three points (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A circle passes through P(13, 1), Q(−11, −9)
and R(6, −16). Find its equation.

Centre is on the perpendicular bisector of each chord.
Bisector of PQ: $-24x - 10y = 16$.
Bisector of QR: $17x - 7y = 45$.
Solving: centre (1, −4).
$r^2 = (13 - 1)^2 + (1 - (-4))^2 = 169$.
Answer: $(x - 1)^2 + (y + 4)^2 = 169$

8 Integration by substitution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use the substitution $u = x^2 + 2$ to find the exact value of

$\int_0^1 x(x^2 + 2)^4\, dx$

$du = 2x\, dx$; limits $x = 0, 1$ become $u = 2, 3$

Integral $= \frac{1}{2}\int_2^3 u^4\, du = [\frac{u^5}{10}]$

Evaluate the limits

Answer: $\frac{211}{10}$

9 Interception at constant speed (AQA, Edexcel, OCR, MEI, Eduqas)

At time t = 0 hours, ship A is at the point with position vector $(32\mathbf{i} - 60\mathbf{j})$ km relative to a port O, and moves with constant velocity $(-3\mathbf{i} + 3\mathbf{j})$ km h^{-1}. Ship B leaves O at t = 0 with constant speed 13 km h^{-1} and intercepts A.
Find (a) the time at which B intercepts A, (b) the velocity of B as a vector.

Position of A at time t: $(32\mathbf{i} - 60\mathbf{j}) + t(-3\mathbf{i} + 3\mathbf{j})$. B at time t: t v_B (unknown velocity)
At interception B has travelled 13 t, so $|(32\mathbf{i} - 60\mathbf{j}) + t(-3\mathbf{i} + 3\mathbf{j})| = 13t$
Squaring and rearranging: $-151t^2 - 552t + 4624 = 0$, which gives t = 4 or a negative value
t = 4 (the other root is negative since B is faster than A)

A is then at $(32\mathbf{i} - 60\mathbf{j}) + 4(-3\mathbf{i} + 3\mathbf{j}) = (20\mathbf{i} - 48\mathbf{j})$, so $v_B = \frac{1}{4}(20\mathbf{i} - 48\mathbf{j}) = (5\mathbf{i} - 12\mathbf{j})$

Answer: (a) 4 h (b) $(5\mathbf{i} - 12\mathbf{j})$ km h^{-1}

10 Closest approach of moving points (Edexcel, OCR, MEI, Eduqas, CIE (vectors in kinematics))

Two drones P and Q move so that, t seconds after launch, their position vectors in metres are
$\mathbf{r}_P = \begin{pmatrix} -5 \\ 3 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$ and $\mathbf{r}_Q = \begin{pmatrix} 2 \\ 0 \\ 9 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ -5 \end{pmatrix}$.
Find the time at which they are closest and the exact minimum distance between them.

$$\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix} 7 \\ -3 \\ 5 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}$$

Distance$^2 = |\mathbf{r}_Q - \mathbf{r}_P|^2$, minimised when $(\mathbf{r}_Q - \mathbf{r}_P) \cdot \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} = 0$

$-27 + t(9) = 0$, so t = 3

Then $\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, with length $\sqrt{2}$

Answer: t = 3, minimum distance = $\sqrt{2}$ m