Day 304 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Trig equations with compound angles (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\cot(2\theta + 40°) = -\sqrt{3}$
for $0° \le \theta \le 360°$.

Let $u = 2\theta + 40°$; then $\cot u = -\sqrt{3}$
Base solutions: $u = -30°$ (+ multiples of 180°)
$\theta = (u - 40°) \div 2$, keeping those in range
Answer: θ = 55°, 145°, 235°, 325°

2 Velocity-time graph with a return (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the velocity-time graph of a particle P moving on a straight line (velocity in m s^{-1}, time in s). At time 16 s, P is back at its starting point.
Find (a) the value of W, (b) the greatest distance of P from its starting point.

Net area = 0 (back at start): $\frac{1}{2} \times 5 \times 6 + \frac{1}{2}(6 - W)(4) - W(7) = 0$
$15 + 12 - 2W - 7W = 0$, so $9W = 27$
$W = 3$

$v = 0$ where the line from (5, 6) to (9, −3) crosses the axis: $t = 5 + 4 \times \frac{6}{9} = \frac{23}{3}$

Greatest distance = area of triangle = $\frac{1}{2} \times \frac{23}{3} \times 6$

Answer: (a) W = 3 (b) 23 m

3 Histogram: frequency density (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The histogram shows the times, t seconds, of some runners.
Find (a) the total number of runners, (b) an estimate of the number with $14 \le t < 34$.

Frequency = frequency density × width. Bar 16 to 26: height 5 squares, width 10, frequency 50
So one square of frequency density = 1 runners per second
Class frequencies = 4, 60, 50, 30, 48
Total = 192
Estimate = 1 × [5(2) + 5(10) + 5(6) + 6(2)] = 102
Answer: (a) 192 (b) 102

4 Binomial expansion (fractional or negative powers) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the binomial expansion of $(27 - 5x)^{\frac{1}{3}}$ in ascending powers of x up to and including the term in x^3.
State the values of x for which it is valid.

$(27 - 5x)^{\frac{1}{3}} = 27^{\frac{1}{3}}(1 - \frac{5}{27}x)^{\frac{1}{3}} = 3(1 + u)^{\frac{1}{3}}$, $u = -\frac{5}{27}x$.

$(1 + u)^{\frac{1}{3}} = 1 + \frac{1}{3}u + (-\frac{1}{9})u^2 + \frac{5}{81}u^3 + ...$

Substitute u and multiply by 3.

Valid when $|u| < 1$, i.e. $|x| < \frac{27}{5}$.

Answer: $3 - \frac{5}{27}x - \frac{25}{2187}x^2 - \frac{625}{531441}x^3 + ...$; $|x| < \frac{27}{5}$

5 Parametric differentiation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = 3t^2 + t$, $y = 3t^3 + 4t$.
Find the equation of the tangent at the point where t = 2,
in the form $y = mx + c$.

$dx/dt = 6t + 1$, $dy/dt = 9t^2 + 4$

At t = 2: $dy/dx = \frac{40}{13} = \frac{40}{13}$; point (14, 32)

$y - 32 = \frac{40}{13}(x - 14)$

Answer: $y = \frac{40}{13}x - \frac{144}{13}$

6 Sums of multiples (arithmetic series) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the sum of all the integers from 1 to 665
inclusive that are divisible by 3 or by 5, but not by both.

Multiples of 3: 221 terms (3 to 663): sum = 73593.
Multiples of 5: 133 terms: sum = 44555.
Multiples of both, i.e. of 15: 44 terms: sum = 14850.
Required sum = 73593 + 44555 − 2 × 14850.
Answer: 88448

7 Boat crossing a river (AQA, Edexcel, OCR, MEI)

A river is 60 m wide and flows at 4 m s^{-1}. A boatman can row at 5 m s^{-1} in still water and wants to reach the point directly opposite his starting point.
Find (a) the angle to the bank at which he should head, to 1 d.p., (b) the time taken, to 3 s.f.

Resultant velocity must be perpendicular to the bank, so the upstream component of the boat's velocity cancels the current
Let the boat head at angle φ upstream from the perpendicular: 5 sin φ = 4, so φ = 53.1°
Angle to the bank = 90° − φ = 36.9°
Speed across = $\sqrt{(5^2 - 4^2)}$ = 3 m s^{-1}
Time = $\frac{60}{3}$ = 20.0 s
Answer: (a) 36.9° (b) 20.0 s

8 Equations with a square root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\sqrt{2x + 5} = 2x - 7$.

Square both sides: $2x + 5 = (2x - 7)^2$.
$4x^2 - 30x + 44 = 0$, so x = 2 or x = $\frac{11}{2}$.
Check in $\sqrt{2x + 5} = 2x - 7$: x = $\frac{11}{2}$ gives $4 \geq 0$ (valid).
x = 2 gives right-hand side $-3 < 0$, so it is rejected.
Answer: x = $\frac{11}{2}$

9 Optimisation with justification (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A closed cylinder has volume 54π cm^3. Find, in terms of π, the
minimum possible total surface area, proving that it is a minimum.

$h = \frac{54}{r^2}$, $S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + \frac{108\pi}{r}$
$dS/dr = 4\pi r - \frac{108\pi}{r^2} = 0$, so $r^3 = 27$, r = 3
$d^2S/dr^2 = 4\pi + \frac{216\pi}{r^3} > 0$, so a minimum
$S = 2\pi \times 9 + \frac{108\pi}{3}$
Answer: 54π cm^2

10 Distribution of the larger of two scores (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Two fair 10-sided dice, numbered 1 to 10, are rolled. M is the smaller of the two scores.

(a) Show that $P(M = m) = \frac{(21 - 2m)}{100}$ for m = 1, 2, ..., 10.

(b) Find P(M = 6) and E(M).

(a) $M \geq m$ means both scores $\geq m$: $P(M \geq m) = (\frac{10 - m + 1}{10})^2$, so $P(M = m) = \frac{(10 - m + 1)^2 - (10 - m)^2}{100} = \frac{(21 - 2m)}{100}$

(b) $P(M = 6) = \frac{9}{100}$

$E(M) = \Sigma\, m\, P(M = m) = \frac{1}{100} \Sigma\, m(21 - 2m)$

$= \frac{77}{20}$

Answer: (b) $P(M = 6) = \frac{9}{100}$, $E(M) = \frac{77}{20}$