Day 303 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Sinusoidal velocity (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves on a straight line. At time t seconds its velocity is v m s^{-1}, where $v = 8\sin(\frac{\pi t}{3})$ for $t \geq 0$.

Find (a) the greatest magnitude of the acceleration of P, (b) the distance P travels between the first two times at which it is at rest. Give (b) to 3 s.f.

$a = \frac{dv}{dt} = \frac{8}{3}\pi\cos(\frac{\pi t}{3})$, so the greatest magnitude is $\frac{8}{3}\pi$

$v = 0$ when $\sin(\frac{\pi t}{3}) = 0$: $t = 0$ and $t = 3$

$s = \int$ from 0 to 3 of $8\sin(\frac{\pi t}{3})\,dt = [-\frac{8 \times 3}{\pi}\cos(\frac{\pi t}{3})]$ from 0 to 3

$= \frac{24}{\pi}(1 - (-1)) = \frac{48}{\pi}$

Answer: (a) $\frac{8}{3}\pi$ m s^{-2} (b) 15.3 m

2 Related rates of change (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A cone with its vertex downwards and semi-vertical angle α, where $\tan\alpha = \frac{1}{2}$, is filled at 9π cm^3s^{-1}. Find the rate at which the depth h is rising when h = 2 cm. ($V = \frac{1}{3}\pi r^2 h$)

$r = \frac{h}{2}$, so $V = \frac{1}{3}\pi \times \frac{h^2}{4} \times h = \frac{\pi}{12}h^3$

$dV/dt = \frac{\pi}{4}h^2\,dh/dt$, so $9\pi = \frac{\pi}{4} \times 4 \times dh/dt$

$dh/dt = \frac{36}{4}$

Answer: 9 cm s^{-1}

3 Resultant of forces: unknown constant and acceleration (AQA, Edexcel, OCR, MEI, Eduqas)

Three forces $(7\mathbf{i} + 2\mathbf{j})$ N, $(-5\mathbf{i} + k\mathbf{j})$ N and $(\mathbf{i})$ N act on a particle of mass 3 kg. The resultant is parallel to $(3\mathbf{i} + 8\mathbf{j})$. Find (a) the value of k, (b) the magnitude of the acceleration of the particle, to 3 s.f.

Resultant = $(3)\mathbf{i} + (2 + k)\mathbf{j}$

Parallel to $(3\mathbf{i} + 8\mathbf{j})$: $\frac{2 + k}{3} = \frac{8}{3}$, so $k = 6$

R = $(3\mathbf{i} + 8\mathbf{j})$ N, so $|R| = \sqrt{(3^2 + 8^2)} = 8.544$

$a = |R| \div 3$

Answer: (a) k = 6 (b) 2.85 m s^{-2}

4 Factor theorem and cubic (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = 3x^3 + ax^2 + bx - 18$, where a and b are constants.
$(x - 3)$ is a factor of f(x), and f(x) leaves remainder -360 when divided by $(x + 3)$.
Find a and b, and hence solve f(x) = 0.

$f(3) = 0$: $81 + 9a + 3b - 18 = 0$, so $9a + 3b + 63 = 0$.
$f(-3) = -360$: $9a - 3b - 99 = -360$.
Solving: $a = -18$, $b = 33$.
$f(x) = 3x^3 - 18x^2 + 33x - 18 = 3(x - 1)(x - 2)(x - 3)$
Answer: a = −18, b = 33; x = 1, 2, 3

5 Regression line from summations (AQA, Edexcel, OCR, MEI)

For n = 8 pairs of values: $\Sigma x = 24$, $\Sigma y = 272$, $\Sigma x^2 = 114$, $\Sigma xy = 942$.
(a) Find the equation of the regression line of y on x.
(b) The x values lie between -1 and 8. Estimate y when x = 7 and comment on its reliability.

$S_{xx} = 114 - \frac{24^2}{8} = 42$

$S_{xy} = 942 - \frac{24 \times 272}{8} = 126$

$b = \frac{S_{xy}}{S_{xx}} = \frac{126}{42} = 3$

$a = \bar{y} - b\,\bar{x} = 34 - (3)(3) = 25$

$y = 25 + 3x$

At x = 7: y = 46. This is extrapolation (outside the data range), so may be unreliable

Answer: (a) y = 25 + 3x (b) 46, unreliable (extrapolation)

6 sec, cosec and tan/cot quadratics (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $4\text{cosec}^2\theta + 10\cot\theta - 10 = 0$
for $0 \leq \theta < 360°$, giving answers to 1 d.p.

Use $\text{cosec}^2\theta \equiv 1 + \cot^2\theta$: $4\cot^2\theta + 10\cot\theta - 6 = 0$

$2(2\cot\theta - 1)(\cot\theta + 3) = 0$

$\cot\theta = \frac{1}{2}$ or $\cot\theta = -3$

$\tan\theta = 2$ or $-\frac{1}{3}$, so $\theta = 63.4°$ and $161.6°$ (+180°)

Answer: θ = 63.4°, 161.6°, 243.4°, 341.6°

7 Sigma notation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Evaluate $\sum_{r=4}^{12}(5 \times 3^r + 3r)$.

Geometric part: first term $5 \times 3^4 = 405$, ratio 3, 9 terms.

Sum = $\frac{405(3^9 - 1)}{2} = 3985605$.

Arithmetic part: 3 × 9 terms, average (4 + 12)/2: sum = 216.

Total = 3985605 + 216.

Answer: 3985821

8 Simultaneous equations (linear and quadratic) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously:

$x + y = -3$

$2x^2 + 2y^2 = 34$

From the first equation $x = -3 - y$.

Substitute into the second equation and collect terms:

this gives $4y^2 + 12y - 16 = 0$.

$y = -4$ or $y = 1$, giving $x = 1$ or $x = -4$.

Answer: (x, y) = (1, −4) and (−4, 1)

9 Normal test: critical region and Type II error (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim N(\mu, 9)$. H_0: $\mu = 85$ is tested against H_1: $\mu < 85$ using the mean of a sample of size 25 at the 10% level.

(a) Find the critical region for the sample mean.

(b) Find the probability of a Type II error when $\mu = 84$.

Under H_0, sample mean ~ $N(85, \frac{9}{25})$, standard error = 0.6000

$z = 1.2816$ for the lower tail: critical value = $85 - 1.2816 \times 0.6000 = 84.23$

Type II error: do not reject H_0, sample mean > 84.23, where mean ~ N(84, 0.6000)

$z = \frac{84.23 - 84}{0.6000} = 0.385$

Probability = 0.350

Answer: (a) sample mean < 84.2 (b) 0.350

10 Partial fractions then integrate (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Express $\frac{-6x - 20}{(x + 2)(x + 4)}$ in partial fractions.

(b) Hence find the exact value of $\int_0^3 \frac{-6x - 20}{(x + 2)(x + 4)}\,dx$.

Let N/D = $\frac{A}{x + 2} + \frac{B}{x + 4}$: $-6x - 20 \equiv A(x + 4) + B(x + 2)$

$x = -2$ gives $A = -4$; $x = -4$ gives $B = -2$

Integral = $[-4 \ln(x + 2) - 2 \ln(x + 4)]$

Substitute limits 0 and 3

Answer: (a) $-4/(x + 2) + (-2)/(x + 4)$; (b) $8\ln 2 - 4\ln 5 - 2\ln 7$