Day 300 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Line tangent to a hyperbola (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $y = \frac{x + 7}{x + 3}$.

(a) Write down the equations of the asymptotes of C.
(b) The line $y = -x + k$ is a tangent to C.
Find the possible values of k.

(a) Vertical: $x = -3$. As $x \to \infty$, $y \to 1$, so $y = 1$.
(b) $x + 7 = (-x + k)(x + 3)$.
$-x^2 + (k - 4)x + (3k - 7) = 0$.
Tangent: $b^2 - 4ac = 0$, which simplifies to $k^2 + 4k - 12 = 0$.
Answer: (a) $x = -3$, $y = 1$ (b) $k = -6$ or $k = 2$

2 Time above a given height (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is thrown vertically upwards from ground level. It is more than 196 m above the ground for 6 s.
Find the speed of projection and the greatest height reached. Take $g = 9.8$ m s^{-2}.

By symmetry, the ball takes 3 s to rise from height 196 m to its highest point
Speed at height 196 m: $v = 9.8 \times 3 = 29.4$ m s^{-1}
Speed of projection: $u^2 = v^2 + 2gh = 864.36 + 2 \times 9.8 \times 196 = 4705.96$
$u = 68.6$. Greatest height $= \frac{u^2}{(2g)} = \frac{4705.96}{19.6} = 240.1$
Answer: 68.6 m s^{-1}, 240.1 m

3 Trapezium rule and over/under estimates (AQA, Edexcel, OCR, MEI, Eduqas (trapezium rule))

Use the trapezium rule with 5 strips to estimate
$\int_0^2 \sqrt{4 - x^2}\, dx$, to 4 s.f.
Is it an overestimate or an underestimate? Explain.

$h = \frac{2}{5}$; ordinates at $x = 0, \frac{2}{5}, \frac{4}{5}, \frac{6}{5}, \ldots, 2$
Sum $= \frac{h}{2}[y_0 + y_5 + 2(y_1 + \ldots + y_4)] = 3.037$
f''(x) < 0 on the interval, so the curve is concave and the trapezia lie below the curve
Answer: 3.037; underestimate

4 Outliers and the 1.5 × IQR rule (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A data set has lower quartile 13 and upper quartile 23. An outlier is a value more than 1.5 × IQR beyond the nearest quartile.
(a) Find the range of values that are not outliers.
(b) 12 observations have mean 23. The largest is 45, an outlier, and it is removed. Find the new mean.

IQR = 23 − 13 = 10, so 1.5 × IQR = 15
Lower limit = 13 − 15 = −2, upper limit = 23 + 15 = 38
$\Sigma x = 12 \times 23 = 276$, so without 45: $\Sigma x = 231$
New mean $= \frac{231}{11}$
Answer: (a) −2 to 38 (b) 21.00

5 Proof by contradiction (logarithms) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove by contradiction that $\log_9 10$ is irrational.

Assume $\log_9 10 = \frac{m}{n}$, m, n positive integers.

Then $9^{\frac{m}{n}} = 10$, so $9^m = 10^n$.
Prime factors of 9^m and 10^n must be identical,
but 2 divides 10^n but 2 does not divide 9^m.
This is a contradiction, so $\log_9 10$ is irrational.
Answer: Proved

6 Large data set: linear change of units (AQA, Edexcel, OCR, MEI, Eduqas)

Large Data Set, daily mean air temperature (°C), Heathrow, October 2015 (n = 92): mean 11.1, sd 3.2. (F = 1.8C + 32)
(a) Find the mean and sd in °F.
(b) Is a reading of 16.9 more than 2 sd above the mean?

New value = 1.8 × old + 32, so mean = 1.8 × 11.1 + 32 = 51.98
The constant shifts the mean but not the spread; only the multiplier affects σ
σ = 1.8 × 3.2 = 5.760
Outlier cut-off = mean + 2σ = 11.1 + 6.4 = 17.5 (original units)
16.9 is less than 17.5, so no
Answer: (a) mean 52.0, sd 5.76 (b) no

7 Implicit differentiation: tangent or normal (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^3 + xy^2 = 5$.
Find the equation of the normal to C at the point (1, −2),
in the form ax + by + c = 0 with integers a, b, c.

Differentiate implicitly with respect to x

At (1, −2): dy/dx = $\frac{7}{4}$

Normal gradient = $-1 \div \frac{7}{4} = -\frac{4}{7}$

$y - -2 = (-\frac{4}{7})(x - 1)$, then clear fractions

Answer: 4x + 7y + 10 = 0

8 Periodic recurrence relations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sequence is defined by $u_1 = -3$ and

$u_{n+1} = \frac{1}{1 - u_n}$.

Show that the sequence is periodic and find

$\sum_{r=1}^{75} u_r$.

$u_1 = -3$, $u_2 = \frac{1}{4}$, $u_3 = \frac{4}{3}$, $u_4 = -3$.

$u_4 = u_1$, so the sequence repeats with period 3.

One period sums to $-\frac{17}{12}$.

75 = 3 × 25 + 0: 25 full periods plus the first 0 terms.

Answer: $-\frac{425}{12}$

9 Direction of motion at 45 degrees (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from level ground with horizontal and vertical velocity components 14.7 and 34.3 m s^{-1}.
Find (a) the two times at which it moves at 45° to the horizontal, (b) the horizontal distance between those two points.
Take g = 9.8 m s^{-2}.

At 45° to the horizontal, |v_y| = v_x = 14.7
Moving upwards: 34.3 − 9.8t = 14.7, so t = 2
Moving downwards: 34.3 − 9.8t = −14.7, so t = 5
Horizontal distance = 14.7 × (5 − 2)
Answer: (a) 2 s and 5 s (b) 44.1 m

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10 R-form: greatest and least values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(\theta) = \frac{2}{32 + 7\sin\theta + 24\cos\theta}$

(a) Write $7\sin\theta + 24\cos\theta$ as $R\sin(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.

(b) Find the greatest and least values of $f(\theta)$, and the smallest positive θ at which the greatest value occurs, to 1 d.p.

$R = \sqrt{7^2 + 24^2} = 25$, $\tan\alpha = \frac{24}{7}$, $\alpha = 73.7°$

Denominator $= 32 + 25\sin(\theta + 73.7°)$, which lies between 7 and 57

Greatest $f(\theta) = \frac{2}{7} = \frac{2}{7}$, when $\sin(\theta + \alpha) = -1$

Least $f(\theta) = \frac{2}{57} = \frac{2}{57}$, when $\sin(\theta + \alpha) = 1$

$\sin(\theta + \alpha) = -1$: $\theta + 73.7° = 270°$, so $\theta = 196.3°$

Answer: (a) $25\sin(\theta + 73.7°)$ (b) greatest $\frac{2}{7}$, least $\frac{2}{57}$, $\theta = 196.3°$