Day 30 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Stone thrown from a cliff (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A stone is thrown vertically upwards with speed 6 m s^{-1} from the top of a cliff. The top of the cliff is 140.4 m above the sea.
Find (a) the time taken for the stone to reach the sea, (b) the total distance travelled by the stone, to 3 s.f. Take g = 9.8 m s^{-2}.

Take upwards as positive: $-140.4 = 6t - 4.9t^2$, so $4.9t^2 - 6t - 140.4 = 0$

$t = \frac{6 + \sqrt{(6^2 + 19.6 \times 140.4)}}{9.8} = \frac{6 + 52.8}{9.8} = 6$

Height risen above the cliff top = $\frac{6^2}{(2 \times 9.8)} = 1.84$ m

Total distance = 2 × 1.84 + 140.4

Answer: (a) 6 s (b) 144 m

2 Surds: rationalising the denominator (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{4 - 3\sqrt{2}}{4 + 2\sqrt{2}}$ in the form $p + q\sqrt{2}$, where p and q are rational numbers.

Multiply top and bottom by the conjugate $(4 - 2\sqrt{2})$.
Numerator: $(4 - 3\sqrt{2})(4 - 2\sqrt{2}) = 28 - 20\sqrt{2}$.
Denominator: $4^2 - 2^2(2) = 8$.
So the fraction = $\frac{28 - 20\sqrt{2}}{8}$.

Answer: $\frac{7}{2} - \frac{5}{2}\sqrt{2}$

3 Time above a given height (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is thrown vertically upwards from ground level with speed 29.4 m s^{-1}.
Find the length of time for which the ball is more than 39.2 m above the ground, and the greatest height it reaches. Take g = 9.8 m s^{-2}.

At height 39.2 m: $v^2 = 29.4^2 - 2 \times 9.8 \times 39.2 = 96.04$, so v = 9.8

Time from this height to the top: $t = \frac{9.8}{9.8} = 1$ s

By symmetry the time above this height is 2 × 1 = 2 s

Greatest height = $\frac{29.4^2}{(2 \times 9.8)} = 44.1$

Answer: 2 s, 44.1 m

4 Outliers and the 1.5 × IQR rule (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A data set has lower quartile 36 and upper quartile 53. An outlier is a value more than 1.5 × IQR beyond the nearest quartile.
(a) Find the range of values that are not outliers.
(b) 12 observations have mean 36. The largest is 90, an outlier, and it is removed. Find the new mean.

IQR = 53 − 36 = 17, so 1.5 × IQR = 25.5
Lower limit = 36 − 25.5 = 10.5, upper limit = 53 + 25.5 = 78.5
$\Sigma x = 12 \times 36 = 432$, so without 90: $\Sigma x = 342$

New mean = $\frac{342}{11}$

Answer: (a) 10.5 to 78.5 (b) 31.09

5 Inverse trigonometric values (Edexcel, OCR, MEI, Eduqas, CIE (inverse trig))

Find the exact value of

$\tan(\tan^{-1}(\frac{1}{9}) + \tan^{-1}(\frac{3}{2}))$.

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$= \frac{\frac{1}{9} + \frac{3}{2}}{1 - \frac{1}{9} \times \frac{3}{2}} = \frac{\frac{29}{18}}{\frac{5}{6}}$

Answer: $\mathbf{\frac{29}{15}}$

6 Compound interest and savings (GP) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each year Jo pays £200 into an account on 1 January.
Interest of 4% is added each 31 December.
Find the first year-end at which the balance exceeds £8,000.

After n years: $200(1.04) + 200(1.04^2) + ... + 200(1.04^n)$.

GP: $a = 200 \times 1.04$, $r = 1.04$: $S_n = \frac{200 \times 1.04\,(1.04^n - 1)}{0.04}$.

$S_n > 8000 \Rightarrow 1.04^n > 2.5385 \Rightarrow n > 23.752$.

So n = 24 years.

Answer: after 24 years

7 Stationary points of trigonometric curves (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact coordinates of the stationary points of
$y = 4x + 8\cos x$ for $0 \le x \le \pi$,
and determine their nature.

$dy/dx = 4 - 8\sin x = 0$, so $\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$

$d^2y/dx^2 = -8\cos x$

Its sign at each point decides maximum or minimum

Answer: $\mathbf{(\frac{\pi}{6}, \frac{2}{3}\pi + 4\sqrt{3})}$ **maximum;** $\mathbf{(\frac{5\pi}{6}, \frac{10}{3}\pi - 4\sqrt{3})}$ **minimum**

8 Proof by exhaustion (modular) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove that $4k = n^2 + 2$ has no integer solutions (k and n integers).

Every integer is 4q, 4q + 1, 4q + 2, 4q + 3 for some integer q.

Squaring, $n^2 \equiv r^2 \pmod 4$, so:

$0 \to 0$; $1 \to 1$; $2 \to 0$; $3 \to 1$ ($r \to n^2 \bmod 4$)

So $n^2 + 2 \equiv 2$ or $3 \pmod 4$.

No residue is 0, so $n^2 + 2$ is never a multiple of 4.

Answer: Proved

9 Poisson: sum of two independent variables, conditional (CIE)

X and Y are independent, X ~ Po(1) and Y ~ Po(4).
(a) State the distribution of X + Y.
(b) Given that X + Y = 5, find the probability that X = 4.

X + Y ~ Po(5)

$P(X = 4 \mid X + Y = 5) = \frac{P(X = 4)P(Y = 1)}{P(X + Y = 5)}$

$= \frac{e^{-1}1^4}{4! \times e^{-4}4^1/1!} \div \frac{e^{-5}5^5}{5!}$

$= C(5, 4)\ (\frac{1}{5})^4\ (\frac{4}{5})^1$, a binomial probability

= 0.00640

Answer: (a) Po(5) (b) 0.00640

10 Separable differential equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$\frac{dy}{dx} = 2xy$, and y = 3 when x = 0.
Solve the differential equation and find y when x = 2.

$\int \frac{1}{y}\, dy = \int 2x\, dx$, so $\ln y = 2x^2/2 + c = 1x^2 + c$

x = 0, y = 3: c = ln 3, so $y = 3\, e^{1x^2}$
x = 2: exponent = 4

Answer: $y = 3\, e^{1x^2}$; $y = 3e^4$