Day 299 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Exponential growth and decay models (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A population of bacteria is modelled by $P = P_0e^{kt}$, t in hours.
Initially P = 200 and after 2 hours P = 500.
Find k, and the time for the population to reach 2000, to 3 s.f.

$P_0 = 200$. $200\,e^{2k} = 500$ so $e^{2k} = \frac{5}{2}$

$k = \frac{1}{2}\ln\frac{5}{2} = 0.458$

$200\,e^{kt} = 2000$ so $e^{kt} = 10$, $t = \frac{1}{k}\ln 10$

t = 5.03 hours

Answer: k = 0.458, t = 5.03 hours

2 Pair of inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the set of values of x for which both
$x^2 - x - 30 \leq 0$ and $-x + 7 \leq 2$.

Quadratic: $(x + 5)(x - 6) \leq 0$ gives $-5 \leq x \leq 6$.
Linear: $-x + 7 \leq 2$ gives $x \geq 5$. (the inequality reverses as we divide by a negative)
Both must hold: take the overlap of the two sets.

Answer: $5 \leq x \leq 6$

3 Resultant of forces: unknown constant and acceleration (AQA, Edexcel, OCR, MEI, Eduqas)

Three forces $(-8\mathbf{i} + 4\mathbf{j})$ N, $(-4\mathbf{i} + k\mathbf{j})$ N and $(-3\mathbf{i} + 2\mathbf{j})$ N act on a particle of mass 4 kg. The resultant is parallel to $(-5\mathbf{i} + \mathbf{j})$.
Find (a) the value of k, (b) the magnitude of the acceleration of the particle, to 3 s.f.

Resultant = $(-15)\mathbf{i} + (6 + k)\mathbf{j}$

Parallel to $(-5\mathbf{i} + \mathbf{j})$: $\frac{6 + k}{-15} = \frac{1}{-5}$, so $k = -3$

$\mathbf{R} = (-15\mathbf{i} + 3\mathbf{j})$ N, so $|R| = \sqrt{(-15^2 + 3^2)} = 15.30$
$a = |R| \div 4$

Answer: (a) $k = -3$ (b) 3.82 m s^{-2}

4 Binomial approximation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Expand $(25 + x)^{-\frac{1}{2}}$ up to and including the term in x^2.
(b) Use x = 0.04 to estimate $\frac{1}{\sqrt{25.04}}$ to 5 decimal places.

(a) $(25 + x)^{-\frac{1}{2}} = \frac{1}{5}(1 + \frac{1}{25}x)^{-\frac{1}{2}}$.

$= \frac{1}{5} - \frac{1}{250}x + \frac{3}{25000}x^2 + ...$

(b) 25 + x with x = 0.04 gives 25.04.

Substitute: $\frac{1}{5} + (-\frac{1}{250})(0.04) + \frac{3}{25000}(0.04)^2$.

= 0.19984 (5 d.p.).

Answer: (a) $\frac{1}{5} - \frac{1}{250}x + \frac{3}{25000}x^2$ (b) 0.19984

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5 Replacing an observation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

12 recorded lengths (cm) have mean 56 and standard deviation 8.
One value was recorded as 56 but should have been 40.
Find the correct mean and the correct standard deviation.

$\Sigma x = 12 \times 56 = 672$, so corrected $\Sigma x = 672 - 56 + 40 = 656$

Correct mean $= \frac{656}{12} = 54.67$

$\Sigma x^2 = 12(64 + 3136) = 38400$, corrected $\Sigma x^2 = 38400 - 3136 + 1600 = 36864$

Variance $= \frac{36864}{12} - 54.667^2 = 83.556$, $\sigma = 9.141$

Answer: mean 54.7, standard deviation 9.14

6 Remainder theorem (quadratic divisor) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the remainder when x^8 is divided by
$(x - 1)(x + 1)$.

$x^8 = (x - 1)(x + 1)Q(x) + px + q$.
$x = 1$: $1 = p + q$.
$x = -1$: $1 = -p + q$.
Subtract: $2p = 0$, so $p = 0$ and $q = 1$.
Answer: 1

7 Stationary points: exact coordinates (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact coordinates of the stationary point of
$y = x^4e^{-x}$ for $x > 0$,
and determine its nature.

$dy/dx = x^3e^{-x}(4 - x) = 0$ gives $x = 0$ or $x = 4$
$y = 256e^{-4}$ when $x = 4$
Second derivative at this point is negative, so it is a maximum
Answer: $(4, 256e^{-4})$, maximum

8 Just clearing a wall at 45 degrees (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground at 45° to the horizontal. The ball just passes over the top of a vertical wall of height 6.8 m, at a horizontal distance of 46 m from O.
Find (a) the speed of projection, (b) the greatest height of the ball above the ground, to 3 s.f. Take $g = 9.8$ m s^{-2}.

At 45°: $x = \frac{u\,t}{\sqrt{2}}$ and $y = \frac{u\,t}{\sqrt{2}} - 4.9t^2$, so $y = x - \frac{9.8x^2}{u^2}$

$6.8 = 46 - \frac{9.8 \times 46^2}{u^2}$, so $u^2 = \frac{9.8 \times 2116}{39.2} = 529$

$u = 23$

Greatest height $= \frac{(u \sin 45°)^2}{(2 \times 9.8)} = \frac{529}{39.2}$

Answer: (a) 23 m s^{-1} (b) 13.5 m

9 Implicit curves: horizontal and vertical tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^2 - 2xy + 5y^2 = 144$.
Find the coordinates of the points on C where the
tangent is parallel to the x-axis.

Differentiate: $2x + by + (bx + 2cy)dy/dx = 0$, so $dy/dx = -\frac{2x + -2y}{-2x + 10y}$

Horizontal tangent when $2x + -2y = 0$, so $y = \frac{2x}{2}$

Substitute into the equation of C: $x^2 + (-2xy) + (5y^2) = 144$
Solve for the remaining coordinate
Answer: $(-6, -6)$ and $(6, 6)$

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10 Normal then binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The mass of a apple is N(117, 5^2) g. A apple is 'heavy' if it exceeds 119.
(a) Find the probability that a apple is heavy.
(b) 8 apples are chosen at random. Find the probability that at least 3 are heavy.

(a) P(X > 119) = P(Z > 0.400) = 0.3446
(b) Let Y = number heavy, Y ~ B(8, 0.3446)
P(Y ≥ 3) = 1 − P(Y ≤ 2) = 0.559
Answer: (a) 0.345 (b) 0.559