Day 298 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Conditions on the roots of a quadratic (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

One root of $x^2 + kx - 8 = 0$ is the square of the other root. Find the possible values of k.

Let the roots be α and α^2.

Product: $\alpha^3 = -8$, so $\alpha = -2$ (the only real cube root).

Sum: $\alpha + \alpha^2 = -2 + 4 = 2 = -k$.

Answer: $k = -2$

2 Without replacement: find the number of counters (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A bag has 6 green counters and some grey counters, n in total.

Two are taken without replacement. P(at least one green) = $\frac{9}{11}$. Find n.

P(neither green) = $\frac{(n-6)}{n} \times \frac{(n-7)}{(n-1)} = \frac{2}{11}$

$\frac{2}{11} n(n-1) = (n-6)(n-7)$

Rearranging gives a quadratic in n whose roots are n = 11 and one other value

The other root is $\frac{14}{3}$ so is rejected (not a valid number of counters)

n = 11

Answer: n = 11

3 Trapezium rule and over/under estimates (AQA, Edexcel, OCR, MEI, Eduqas (trapezium rule))

Use the trapezium rule with 4 strips to estimate

$\int_0^1 x\, e^{-x}\, dx$, to 4 s.f.

Is it an overestimate or an underestimate? Explain.

h = $\frac{1}{4}$; ordinates at x = 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, …, 1

Sum = $\frac{h}{2}[y_0 + y_4 + 2(y_1 + \ldots + y_3)] = 0.2590$

f''(x) < 0 on the interval, so the curve is concave and the trapezia lie below the curve

Answer: 0.2590; underestimate

4 Surds: rationalising the denominator (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{-1 + 2\sqrt{7}}{1 - \sqrt{7}}$ in the form $p + q\sqrt{7}$, where p and q are rational numbers.

Multiply top and bottom by the conjugate $(1 + \sqrt{7})$.

Numerator: $(-1 + 2\sqrt{7})(1 + \sqrt{7}) = 13 + \sqrt{7}$.

Denominator: $1^2 - 1^2(7) = -6$.

So the fraction = $\frac{13 + \sqrt{7}}{-6}$.

Answer: $-\frac{13}{6} - \frac{1}{6}\sqrt{7}$

5 Newton cooling model (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A drink cools according to $\theta = 15 + Ae^{-kt}$, where θ °C is the temperature after t minutes. Initially $\theta = 85$, and after 10 minutes $\theta = 50$.
Find A and k, then find when $\theta = 43$ (3 s.f.).

$t = 0$: $15 + A = 85$ so $A = 70$

$t = 10$: $15 + 70e^{-10k} = 50$ so $e^{-10k} = \frac{1}{2}$

$k = \frac{1}{10} \ln 2 = 0.0693$

$70\, e^{-kt} = 28$ so $t = \frac{1}{k} \ln \frac{5}{2}$

$t = 13.2$ minutes

Answer: A = 70, k = 0.0693, t = 13.2 minutes

6 Rough table and smooth slope: string cut (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particle A (4 kg) lies on a rough table with $\mu = 0.25$, and is joined by a light string over a smooth pulley (see diagram) to particle B (4 kg) on a smooth slope, $\sin \alpha = \frac{4}{5}$.
Released from rest, the string is cut when B has moved 1.5 m. Find the further distance A slides before stopping.

A: $T - 0.25 \times 4 \times 9.8 = 4a$. B: $4 \times 9.8 \times \frac{4}{5} - T = 4a$

Adding: $9.8(4 \times \frac{4}{5} - 0.25 \times 4) = 8a$, so $a = 2.695$

$T = 4(2.695 + 0.25 \times 9.8) = 20.58$

When cut: $v^2 = 2 \times 2.695 \times 1.5 = 8.085$. Then A decelerates at $\mu g = 2.45$

Further distance $= \frac{8.085}{2 \times 2.45} = 1.65$

Answer: 1.65 m

7 Stationary points of x^n times a root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = x^3\sqrt{175 - x}$, for $0 < x < 175$, has one stationary point. Find its exact coordinates and determine its nature.

$\frac{dy}{dx} = \frac{x^2(1050 - 7x)}{2\sqrt{175 - x}}$

$\frac{dy}{dx} = 0$: $1050 - 7x = 0$, so $x = 150$

$y = 150^3 \times 5$

Gradient changes from positive to negative: a maximum

Answer: (150, 16875000), maximum

8 Displacement of the form R sin(ωt + φ) (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves on a line. Its displacement from a fixed point O at time t seconds is s metres, where $s = 6 \sin t + 8 \cos t$.
Find (a) the greatest distance of P from O, (b) the greatest speed of P, (c) the first time P is at its greatest distance from O (3 s.f.).

Write $s = R \sin(t + \varphi)$ with $R \cos \varphi = 6$ and $R \sin \varphi = 8$

$R = \sqrt{(6^2 + 8^2)} = 10$ and $\tan \varphi = \frac{8}{6}$, so $\varphi = 0.9273$ rad

Greatest distance = R = 10

$v = \frac{ds}{dt} = 10 \times 1 \cos(t + \varphi)$, so greatest speed = 10

s is greatest when $t + \varphi = \frac{\pi}{2}$: $t = (1.5708 - 0.9273) \div 1$

Answer: (a) 10 m (b) 10 m s^{-1} (c) 0.644 s

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9 Arithmetic terms forming a geometric sequence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The first term of an arithmetic sequence is 4 and its common difference d is non-zero. The 3rd, 7th and 15th terms are the first three terms of a geometric sequence. Find d and the common ratio.

Terms: $4 + 2d$, $4 + 6d$, $4 + 14d$.

Geometric: $(4 + 6d)^2 = (4 + 2d)(4 + 14d)$.

Expand: $36d^2 + 48d + 16 = 28d^2 + 64d + 16$, so $d(8d - 16) = 0$.

$d \neq 0$, so $d = 2$. Ratio $= \frac{16}{8} = 2$.

Answer: d = 2, ratio = 2

10 Cumulative distribution function (CIE)

The cdf of X is $F(x) = 0$ for $x < 0$, $F(x) = 1 - (1 - \frac{x}{4})^4$ for $0 \leq x \leq 4$, and $F(x) = 1$ for $x > 4$.

(a) Find the pdf of X.
(b) Find E(X).
(c) Find P(1 < X < 2) and the median, to 3 significant figures.

(a) $f(x) = F'(x) = \frac{4}{4}(1 - \frac{x}{4})^3$ for $0 \leq x \leq 4$

(b) $E(X) = \int x\, f(x)\, dx$ from 0 to 4 $= \frac{4}{5}$

(c) $P = F(2) - F(1) = 0.2539$

Median m: $F(m) = 1/2$, so $m = 4(1 - (1/2)^{1/4})$

$m = 0.6364$

Answer: (a) $f(x) = \frac{4}{4}(1 - \frac{x}{4})^3$ (b) $\frac{4}{5}$ (c) 0.254, median 0.636