Day 297 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Expected winnings and a fair game (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A bag has 4 red, 3 blue and 2 green counters. A player takes two without replacement. Two reds win £3, two blues win £9, anything else wins nothing.
(a) Find the expected winnings.
(b) The player pays £1 to play. Find the expected profit.

$P(RR) = \frac{4}{9} \times \frac{3}{8} = \frac{1}{6}$

$P(BB) = \frac{3}{9} \times \frac{2}{8} = \frac{1}{12}$

(a) $E = 3 \times \frac{1}{6} + 9 \times \frac{1}{12} = \frac{5}{4}$

(b) Expected profit $= \frac{5}{4} - 1 = \frac{1}{4}$

Answer: (a) £1.25 (b) £0.250

2 Equations quadratic in an exponential (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $3^{2x} - 3 \times 3^x + 2 = 0$,
giving each answer exactly and, where not an integer, to 3 s.f.

Let $u = 3^x$: $u^2 - 3u + 2 = 0$
$(u - 1)(u - 2) = 0$, so $3^x = 1$ or $3^x = 2$
Take logarithms: $x = \log_3 1 = 0$; $x = \log_3 2 = 0.631$
Answer: $x = \log_3 1 = 0$; $x = \log_3 2 = 0.631$

3 Sigma notation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Evaluate $\sum_{r=3}^{10}(4 \times 2^r + 2r)$.

Geometric part: first term $4 \times 2^3 = 32$, ratio 2, 8 terms.
Sum $= \frac{32(2^8 - 1)}{1} = 8160$.
Arithmetic part: 2×8 terms, average $(3 + 10)/2$: sum = 104.
Total = 8160 + 104.
Answer: 8264

4 Newton cooling model (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A drink cools according to $\theta = 15 + Ae^{-kt}$, where θ °C is the temperature after t minutes. Initially $\theta = 90$, and after 4 minutes $\theta = 60$.
Find A and k, then find when $\theta = 30$ (3 s.f.).

t = 0: $15 + A = 90$ so $A = 75$

t = 4: $15 + 75e^{-4k} = 60$ so $e^{-4k} = \frac{3}{5}$

$k = \frac{1}{4} \ln \frac{5}{3} = 0.128$

$75\,e^{-kt} = 15$ so $t = \frac{1}{k} \ln 5$

t = 12.6 minutes
Answer: A = 75, k = 0.128, t = 12.6 minutes

5 Indices: quadratic in a^x (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $5^{2x} + 3 \times 5^x - 4 = 0$.

Let $y = 5^x$: $y^2 + 3y - 4 = 0$.
$(y - 1)(y + 4) = 0$, so $y = 1$ or $y = -4$.
$5^x > 0$ for all x, so $y = -4$ is rejected.
$5^x = 1$ gives $x = 0$.
Answer: x = 0

6 Implicit differentiation: tangent or normal (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x e^y + y = 3$.
Find the equation of the tangent to C at the point (3, 0),
in the form $ax + by + c = 0$ with integers a, b, c.

Differentiate implicitly with respect to x

At (3, 0): $dy/dx = -\frac{1}{4}$

$y - 0 = (-\frac{1}{4})(x - 3)$, then clear fractions

Answer: $x + 4y - 3 = 0$

7 Closest approach of two ships (AQA, Edexcel, OCR, MEI, Eduqas)

Two ships A and B move with constant velocities. At time t = 0 hours, A is at $(-4\mathbf{i} + 3\mathbf{j})$ km relative to a fixed origin and B is at $(-26\mathbf{i} - 18\mathbf{j})$ km. A has velocity $(-5\mathbf{i} - 2\mathbf{j})$ km h^{-1} and B has velocity $(\mathbf{i} + 6\mathbf{j})$ km h^{-1}.
Find (a) the time at which the ships are closest together, (b) the least distance between them.

Position of B relative to A: $\mathbf{d} = (-22\mathbf{i} - 21\mathbf{j}) + t(6\mathbf{i} + 8\mathbf{j})$

$|\mathbf{d}|^2 = (6t - 22)^2 + (8t - 21)^2 = 100t^2 - 600t + 925$

Least when $\frac{d}{dt}|\mathbf{d}|^2 = 0$: $t = -\frac{-300}{100} = 3$

$\mathbf{d} = (-22\mathbf{i} - 21\mathbf{j}) + 3(6\mathbf{i} + 8\mathbf{j}) = (-4\mathbf{i} + 3\mathbf{j})$, so least distance = 5

Answer: (a) t = 3 h (b) 5 km

8 Projectile through a given point: two angles (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground at 21 m s^{-1}, at angle θ above the horizontal. It passes through the point P, 18 m horizontally and 14 m vertically from O.
Find the two possible values of $\tan\theta$, and the two possible values of θ to 1 d.p. Take g = 9.8 m s^{-2}.

Horizontal: $x = 21\cos\theta \times t$. Vertical: $y = 21\sin\theta \times t - 4.9t^2$

Eliminating t: $y = x\tan\theta - \frac{9.8x^2}{2u^2}(1 + \tan^2\theta)$

$\frac{9.8 \times 18^2}{2 \times 21^2} = 3.6$, so $14 = 18\tan\theta - 3.6(1 + \tan^2\theta)$

$3.6\tan^2\theta - 18\tan\theta + 17.6 = 0$

$\tan\theta = \frac{4}{3}$ or $\frac{11}{3}$, so $\theta = 53.1°$ or $74.7°$

Answer: $\tan\theta = \frac{4}{3}$ or $\frac{11}{3}$; $\theta = 53.1°$ or $74.7°$

9 Rejected batches: binomial inside binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box holds 7 components, each faulty with probability 0.05, independently. A box is rejected if it has at least 2 faulty components.
(a) Find the probability that a box is rejected.
(b) 14 boxes are inspected. Find the probability that at least 3 are rejected.

Let $X \sim B(7, 0.05)$. $P(\text{rejected}) = P(X \geq 2) = 1 - P(X \leq 1) = 0.04438$
Let Y = number of rejected boxes, $Y \sim B(14, 0.04438)$
$P(Y \geq 3) = 1 - P(Y \leq 2) = 0.0220$
Answer: (a) 0.0444 (b) 0.0220

10 Parametric curve and line intersection (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has parametric equations
$x = t$, $y = 3t^2$. The line $y = 15x - 12$
meets C at A and B. Find the exact length AB.

Substitute: $3t^2 = 15t - 12$.
$3t^2 - 15t + 12 = 0 \Rightarrow t = 1$ or $t = 4$.
A(1, 3), B(4, 48).
$AB^2 = (-3)^2 + (-45)^2 = 2034$.
Answer: AB = $3\sqrt{226}$