Day 294 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Tangent line to circle: finding k (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The line $8y = 6x + k$ is a tangent to the circle $(x + 3)^2 + (y + 1)^2 = 4$.
Find the two possible values of k.

Substitute $y = \frac{6x + k}{8}$: $(x + 3)^2 + (\frac{6x + k}{8} + 1)^2 = 4$.

Multiply by 64 and collect: a quadratic in x with a = 100.
Tangent ⇒ discriminant = 0. This simplifies to $(k - 10)^2 = 400$.
$k - 10 = \pm 20$.
Answer: $k = -10$ or $k = 30$

2 Inverse trigonometric values (Edexcel, OCR, MEI, Eduqas, CIE (inverse trig))

Find the exact value of $\tan(2\cos^{-1}(\frac{21}{29}))$.

Let $A = \cos^{-1}(\frac{21}{29})$: $\cos A = \frac{21}{29}$, $\sin A = \frac{20}{29}$, $\tan A = \frac{20}{21}$

$\tan 2A = \frac{2\tan A}{1 - \tan^2 A} = \frac{\frac{40}{21}}{\frac{41}{441}}$

Answer: $\frac{840}{41}$

3 Arithmetic series modelling (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A pile of logs has 19 logs in its bottom row. Each row above has 1 fewer logs than the row below, and the pile has as many rows as possible, each with at least one log. Find the number of rows and the total number of logs.

Row n has $19 - 1(n - 1) = -n + 20$ logs; need this ≥ 1.
$1(n - 1) \leq 18$, so $n \leq 19$: 19 rows.

Top row = 1 logs. $S = \frac{19}{2}(19 + 1) = 190$.

Answer: 19 rows, 190 logs

4 Newton cooling model (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A drink cools according to $\theta = 20 + Ae^{-kt}$, where θ °C is the temperature after t minutes. Initially $\theta = 100$, and after 10 minutes $\theta = 68$.
Find A and k, then find when $\theta = 43$ (3 s.f.).

$t = 0$: $20 + A = 100$ so $A = 80$

$t = 10$: $20 + 80e^{-10k} = 68$ so $e^{-10k} = \frac{3}{5}$

$k = \frac{1}{10} \ln \frac{5}{3} = 0.0511$

$80\,e^{-kt} = 23$ so $t = \frac{1}{k} \ln \frac{80}{23}$

$t = 24.4$ minutes
Answer: A = 80, k = 0.0511, t = 24.4 minutes

5 Rejected batches: binomial inside binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box holds 8 components, each faulty with probability 0.15, independently. A box is rejected if it has at least 2 faulty components.
(a) Find the probability that a box is rejected.
(b) 12 boxes are inspected. Find the probability that at least 2 are rejected.

Let $X \sim B(8, 0.15)$. $P(\text{rejected}) = P(X \geq 2) = 1 - P(X \leq 1) = 0.3428$
Let Y = number of rejected boxes, $Y \sim B(12, 0.3428)$
$P(Y \geq 2) = 1 - P(Y \leq 1) = 0.953$
Answer: (a) 0.343 (b) 0.953

6 Parametric differentiation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = t^2$, $y = 2t^3 + 36t$.
Find the coordinates of the points on the curve where the gradient is 15.

$dx/dt = 2t$, $dy/dt = 6t^2 + 36$, so $dy/dx = \frac{6t^2 + 36}{2t}$

Set equal to 15: $6t^2 - 30t + 36 = 0$
$t = 2$ or $t = 3$
Answer: (4, 88) and (9, 162)

7 Normal: unknown standard deviation then a second probability (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The masses of apples, in grams are modelled by $N(69, \sigma^2)$. The probability of a value above 78 is 0.15.
(a) Find σ.
(b) Find the probability of a value below 60.

$P(X > 78) = 0.15$ so $Z = \frac{78 - 69}{\sigma} = 1.0364$

$\sigma = \frac{9}{1.0364} = 8.684$

$z = \frac{60 - 69}{8.684} = -1.036$

$P(X < 60) = P(Z < -1.036) = 0.150$
Answer: (a) 8.68 (b) 0.150

8 Block on a plank: limiting slip (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A plank of mass 5 kg lies on a smooth horizontal table, with a block of mass 3 kg on it ($\mu = 0.5$ between them). A horizontal force P N acts on the block.
(a) Find the greatest P for which the block does not slip. (b) Find the accelerations of block and plank when P = 70.56.
Take $g = 9.8$ m s^{-2}.

No slipping: common acceleration $a = \frac{P}{8}$. The plank is accelerated only by friction: $F = 5a$

Limiting friction: $F = \mu mg = 0.5 \times 3 \times 9.8 = 14.7$, so $a = \frac{14.7}{5} = 2.94$

$P_{max} = 8 \times 2.94 = 23.52$

$P = 70.56 > 23.52$, so the block slips. Plank: $a = \frac{14.7}{5} = 2.94$

Block: $70.56 - 14.7 = 3a$, so $a = 18.62$
Answer: (a) 23.52 N (b) block 18.62 m s^{-2}, plank 2.94 m s^{-2}

9 Simultaneous equations (linear and quadratic) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously:
$x - y = 5$
$x^2 + 2xy - y^2 = -7$

From the first equation $x = 5 + y$.
Substitute into the second equation and collect terms:
this gives $2y^2 + 20y + 32 = 0$.
$y = -8$ or $y = -2$, giving $x = -3$ or $x = 3$.
Answer: (x, y) = (−3, −8) and (3, −2)

10 Closest approach of two ships (AQA, Edexcel, OCR, MEI, Eduqas)

Two ships A and B move with constant velocities. At time t = 0 hours, A is at $(9\mathbf{i} - 8\mathbf{j})$ km relative to a fixed origin and B is at $(19\mathbf{i} - 13\mathbf{j})$ km. A has velocity $(2\mathbf{i} + 3\mathbf{j})$ km h^{-1} and B has velocity $(-\mathbf{i} + 7\mathbf{j})$ km h^{-1}.
Find (a) the time at which the ships are closest together, (b) the least distance between them.

Position of B relative to A: $\mathbf{d} = (10\mathbf{i} - 5\mathbf{j}) + t(-3\mathbf{i} + 4\mathbf{j})$

$|\mathbf{d}|^2 = (-3t + 10)^2 + (4t - 5)^2 = 25t^2 - 100t + 125$

Least when $\frac{d}{dt}|\mathbf{d}|^2 = 0$: $t = -\frac{-50}{25} = 2$

$\mathbf{d} = (10\mathbf{i} - 5\mathbf{j}) + 2(-3\mathbf{i} + 4\mathbf{j}) = (4\mathbf{i} + 3\mathbf{j})$, so least distance = 5

Answer: (a) t = 2 h (b) 5 km