Day 292 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Surds: nested radicals (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\sqrt{16 - 6\sqrt{7}}$ in the form
$a - b\sqrt{7}$, where a and b are positive integers.

Suppose $\sqrt{16 - 6\sqrt{7}} = a - b\sqrt{7}$. Square both sides.
$16 - 6\sqrt{7} = a^2 + 7b^2 - 2ab\sqrt{7}$.
Equate: $a^2 + 7b^2 = 16$ and $2ab = 6$, so $ab = 3$.
Try $a = 3$, $b = 1$: $9 + 7 = 16$ ☐.
($a > b\sqrt{7}$, so the root is positive.)
Answer: $3 - 1\sqrt{7}$

2 Just clearing a wall at 45 degrees (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground at 45° to the horizontal. The ball just passes over the top of a vertical wall of height 28.8 m, at a horizontal distance of 80 m from O.
Find (a) the speed of projection, (b) the greatest height of the ball above the ground, to 3 s.f. Take $g = 9.8$ m s^{-2}.

At 45°: $x = \frac{u\,t}{\sqrt{2}}$ and $y = \frac{u\,t}{\sqrt{2}} - 4.9t^2$, so $y = x - \frac{9.8x^2}{u^2}$

$28.8 = 80 - \frac{9.8 \times 80^2}{u^2}$, so $u^2 = \frac{9.8 \times 6400}{51.2} = 1225$

$u = 35$

Greatest height $= \frac{(u \sin 45°)^2}{(2 \times 9.8)} = \frac{1225}{39.2}$

Answer: (a) 35 m s^{-1} (b) 31.2 m

3 Composite functions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = -x + 2$ and $g(x) = x^2 - 1$.
Solve $fg(x) = gf(x)$.

$fg(x) = -(x^2 - 1) + 2 = -x^2 + 3$.
$gf(x) = (-x + 2)^2 - 1 = x^2 - 4x + 3$.
Equate: $2x^2 - 4x = 0$.
Solve the quadratic: $x = 0$ or $x = 2$.
Answer: $x = 0$ or $x = 2$

4 Independence from a Venn diagram (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The Venn diagram shows probabilities; A and B are independent.
Find x and y.

$P(A) = 2x + x = 3x$, $P(B) = x + y$, $P(A \cap B) = x$

Independent: $x = 3x(x + y)$, and $x \neq 0$, so $x + y = \frac{1}{3}$

Total probability: $2x + x + y + 0.05 = 1$, so $2x + (x + y) = 0.95$

$2x = 0.95 - \frac{1}{3} = \frac{37}{60}$

$x = \frac{37}{120}$, $y = \frac{1}{3} - x = \frac{1}{40}$

Answer: $x = \frac{37}{120}$, $y = \frac{1}{40}$

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5 Segments: exact area and perimeter (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A chord AB of a circle, centre O and radius 8 cm, subtends an angle of $\frac{\pi}{2}$ at O.

Find the exact area and the exact perimeter of the minor segment cut off by AB.

Sector OAB: $\frac{1}{2}r^2\theta = \frac{1}{2} \times 8^2 \times \frac{\pi}{2}$; triangle OAB: $\frac{1}{2}r^2\sin\theta = 32$

Area = $16\pi - 32$

Chord AB = $2r\sin(\theta/2) = 8\sqrt{2}$

Arc AB = $r\theta = 4\pi$

Perimeter = arc + chord

Answer: Area = $-32 + 16\pi$ cm^2, perimeter = $4\pi + 8\sqrt{2}$ cm

6 Indices: quadratic in a^x (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $2^{2x} - 5 \times 2^x + 4 = 0$.

Let $y = 2^x$. Then $y^2 - 5y + 4 = 0$.

$(y - 1)(y - 4) = 0$, so $y = 1$ or $y = 4$.

$2^x = 1$ gives $x = 0$; $2^x = 4$ gives $x = 2$.

Answer: x = 0 or x = 2

7 Parametric differentiation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = \ln t$, $y = t^2 - 4t$, $t > 0$.

Find the equation of the tangent where $t = 1$, in the form $y = mx + c$.

$dx/dt = \frac{1}{t}$, $dy/dt = 2t - 4$

$dy/dx = t(2t - 4)$ which is -2 at $t = 1$

At $t = 1$: $x = 0$, $y = -3$

Answer: $y = -2x - 3$

8 Rough slope: up then back down (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle is projected at 9 m s^{-1} up a line of greatest slope of a rough plane inclined at α to the horizontal, where $\sin\alpha = \frac{4}{5}$ and $\mu = 0.95$.

Find (a) the distance it travels up the plane, to 3 s.f., (b) its speed on returning to its starting point. Take $g = 9.8$ m s^{-2}.

$\cos\alpha = \frac{3}{5}$. Up the plane: $R = mg\cos\alpha$, so $a_{up} = g(\sin\alpha + \mu\cos\alpha) = 9.8(\frac{4}{5} + 0.95 \times \frac{3}{5}) = 13.426$

$0 = 9^2 - 2 \times 13.426 \times d$, so $d = \frac{81}{26.852} = 3.017$

Down the plane: $a_{down} = g(\sin\alpha - \mu\cos\alpha) = 9.8(\frac{4}{5} - 0.95 \times \frac{3}{5}) = 2.254$

$v^2 = 2 \times 2.254 \times 3.0165 = 13.60$, so $v = 3.69$

Answer: (a) 3.02 m (b) 3.69 m s^{-1}

9 Separable differential equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$\frac{dy}{dx} = \frac{6x^2}{y}$, $y > 0$, and $y = 7$ when $x = 0$.

Find y in terms of x, and the value of y when $x = 2$.

$\int y\,dy = \int 6x^2\,dx$ gives $\frac{y^2}{2} = 2x^3 + c$

$x = 0$, $y = 7$: $c = \frac{49}{2}$, so $y^2 = 4x^3 + 49$

$x = 2$: $y^2 = 81$

Answer: $y = \sqrt{4x^3 + 49}$; $y = 9$

10 Expected winnings and a fair game (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A game costs £1. Two fair dice are rolled: a total of at least 12 wins £14, a total of at most 4 wins £2, otherwise nothing.
(a) Find the expected profit.
(b) Find the standard deviation of the profit (3 s.f.).
(c) Find the stake that makes the game fair.

P(total ≥ 12) = $\frac{1}{36}$, P(total ≤ 4) = $\frac{1}{6}$, otherwise $\frac{29}{36}$

Profit: £13 with $\frac{1}{36}$, £1 with $\frac{1}{6}$, £-1 with $\frac{29}{36}$

(a) E = $-\frac{5}{18}$

(b) E(P^2) = $\frac{17}{3}$, Var = $\frac{17}{3} - (-\frac{5}{18})^2 = \frac{1811}{324}$, so $\sigma = 2.364$

(c) Fair when the stake equals E(winnings) = $\frac{1}{36} \times 14 + \frac{1}{6} \times 2 = \frac{13}{18}$

Answer: (a) $-£\frac{5}{18}$ (b) 2.36 (c) $£\frac{13}{18}$