Day 290 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Surds: rationalising the denominator (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Show that $\frac{6+\sqrt{5}}{6-\sqrt{5}} - \frac{6-\sqrt{5}}{6+\sqrt{5}}$

can be written in the form $k\sqrt{5}$, and find k.

Common denominator $(6-\sqrt{5})(6+\sqrt{5}) = 36 - 5 = 31$.

Numerator: $(6+\sqrt{5})^2 - (6-\sqrt{5})^2$

$= (36 + 5 + 26\sqrt{5}) - (36 + 5 - 26\sqrt{5}) = 24\sqrt{5}$.

So the expression $= \frac{24\sqrt{5}}{31}$.

Answer: k = $\frac{24}{31}$

2 Stone thrown from a cliff (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A stone is thrown vertically upwards with speed 8 m s^{-1} from the top of a cliff. The top of the cliff is 82.5 m above the sea. Find (a) the time taken for the stone to reach the sea, (b) the total distance travelled by the stone, to 3 s.f. Take g = 9.8 m s^{-2}.

Take upwards as positive: $-82.5 = 8t - 4.9t^2$, so $4.9t^2 - 8t - 82.5 = 0$

$t = \frac{8 + \sqrt{(8^2 + 19.6 \times 82.5)}}{9.8} = \frac{8 + 41}{9.8} = 5$

Height risen above the cliff top $= \frac{8^2}{(2 \times 9.8)} = 3.27$ m

Total distance $= 2 \times 3.27 + 82.5$

Answer: (a) 5 s (b) 89.0 m

3 Simultaneous logarithmic equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve the simultaneous equations

$\log_2 x + \log_2 y = 6$

$x - y = 12$

$\log_2(xy) = 6$, so $xy = 2^6 = 64$

$x = y + 12$: $y^2 + 12y - 64 = 0$

$(y - 4)(y + 16) = 0$; $y > 0$ so $y = 4$

Then $x = y + 12$

Answer: x = 16, y = 4

4 Direction of motion at 45 degrees (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from level ground with horizontal and vertical velocity components 49 and 53.9 m s^{-1}.

Find (a) the two times at which it moves at 45° to the horizontal, (b) the horizontal distance between those two points.

Take g = 9.8 m s^{-2}.

At 45° to the horizontal, $|v_y| = v_x = 49$

Moving upwards: $53.9 - 9.8t = 49$, so $t = 0.5$

Moving downwards: $53.9 - 9.8t = -49$, so $t = 10.5$

Horizontal distance $= 49 \times (10.5 - 0.5)$

Answer: (a) 0.5 s and 10.5 s (b) 490 m

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5 Range and domain by completing the square (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{5}{2x^2 - 16x + 38}$ for all real x.

Find the greatest value of f(x), the value of x at which it occurs, and the range of f.

$2x^2 - 16x + 38 = 2(x - 4)^2 + 6$.

$(x - 4)^2 \geq 0$, so the denominator is at least 6, with equality at x = 4.

The denominator is always positive, so f(x) > 0.

Greatest f = $\frac{5}{6} = \frac{5}{6}$ at x = 4.

Answer: greatest value $\frac{5}{6}$ at x = 4; range $0 < f(x) \leq \frac{5}{6}$

6 Without replacement: find the number of counters (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A bag has 6 yellow counters and some white counters, n in total.

Two are taken without replacement. P(both yellow) = $\frac{15}{91}$. Find n.

P(both yellow) = $\frac{6}{n} \times \frac{5}{(n-1)} = \frac{30}{n(n-1)}$

So $n(n - 1) = 30 \div \frac{15}{91} = 182$

$n^2 - n - 182 = 0$, so $(n - 14)(n + 13) = 0$

n > 0, so n = 14

Answer: n = 14

7 Integration by parts (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use integration by parts to find the exact value of

$\int_1^3 x \ln x \, dx$

Let u = ln x and dv/dx = x, so du/dx = $\frac{1}{x}$, v = $\frac{x^2}{2}$

Integral = $[\frac{x^2}{2} \ln x] - \int \frac{x^1}{2} dx$

$= [\frac{x^2}{2} \ln x - \frac{x^2}{4}]$

Answer: $-2 + \frac{9}{2}\ln 3$

8 Independence from a Venn diagram (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The Venn diagram shows probabilities; A and B are independent.

Find the possible values of x.

P(A) = 0.15 + x, P(B) = 0.2 + x, $P(A \cap B) = x$

Independent: $x = (0.15 + x)(0.2 + x)$

$x^2 - 0.65x + 0.03 = 0$

$(x - 0.05)(x - 0.6) = 0$

Both values keep every probability between 0 and 1

Answer: x = 0.05 or x = 0.6

9 Cubic graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the cubic y = f(x), which touches the x-axis once. Find f(x), giving your answer in expanded form.

Crosses at x = 3 and touches at x = −2, so $f(x) = k(x - 3)(x + 2)^2$.

At (0, 12): $12 = k(-3)(2)^2 = -12k$, so k = −1.

$f(x) = -(x - 3)(x + 2)^2$, then expand.

Answer: $f(x) = -x^3 - x^2 + 8x + 12$

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10 Separable differential equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$\frac{dy}{dx} = 6xy$, and $y = 2$ when $x = 0$.

Solve the differential equation and find y when $x = 2$.

$\int \frac{1}{y}\, dy = \int 6x\, dx$, so $\ln y = 6x^2/2 + c = 3x^2 + c$

$x = 0$, $y = 2$: $c = \ln 2$, so $y = 2\, e^{3x^2}$

$x = 2$: exponent = 12

Answer: $y = 2\, e^{3x^2}$; $y = 2e^{12}$