Day 29 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Composite functions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$g(x) = x^2 + 5$. Also $fg(x) = -3x^2 - 13$, where f is a linear function. (a) Find f(x).
(b) Solve $gf(x) = 30$.

(a) Let $f(x) = ax + b$. $fg(x) = a(x^2 + 5) + b = -3x^2 - 13$.
So $a = -3$ and $-15 + b = -13$, giving $b = 2$.
$f(x) = -3x + 2$.
(b) $gf(x) = (-3x + 2)^2 + 5 = 30$.
$(-3x + 2)^2 = 25$, so $-3x + 2 = \pm 5$.
$x = -1$ or $x = \frac{7}{3}$.

Answer: (a) f(x) = −3x + 2 (b) x = −1 or x = $\frac{7}{3}$

2 Equations quadratic in an exponential (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $2^{2x} - 21 \times 2^x + 80 = 0$,
giving each answer exactly and, where not an integer, to 3 s.f.

Let $u = 2^x$: $u^2 - 21u + 80 = 0$
$(u - 16)(u - 5) = 0$, so $2^x = 16$ or $2^x = 5$
Take logarithms: $x = \log_2 5 = 2.32$; $x = \log_2 16 = 4$
Answer: x = $\log_2$ 5 = 2.32; x = $\log_2$ 16 = 4

3 Equations with a square root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\sqrt{3x + 1} = 3x - 1$.

Square both sides: $3x + 1 = (3x - 1)^2$.
$9x^2 - 9x = 0$, so $x = 0$ or $x = 1$.
Check in $\sqrt{3x + 1} = 3x - 1$: $x = 1$ gives $2 \geq 0$ (valid).
$x = 0$ gives right-hand side $-1 < 0$, so it is rejected.
Answer: x = 1

4 Identity then equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Show that $\cos^4\theta - \sin^4\theta \equiv \cos 2\theta$.

(b) Hence solve $\cos^4\theta - \sin^4\theta = \frac{1}{2}$ for $0 \leq \theta < 360°$ (1 d.p.).

$\cos^4\theta - \sin^4\theta = (\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta) = \cos 2\theta \times 1$

$\cos 2\theta = \frac{1}{2}$, so $2\theta = 60.0°, 300.0°, 420.0°, 660.0°$ $(0 \leq 2\theta < 720°)$

Halve each value
Answer: θ = 30.0°, 150.0°, 210.0°, 330.0°

5 Pair of inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the set of values of x for which both
$x^2 + 4x > 0$ and $-2x + 8 \geq 11$.

Quadratic: $(x + 4)(x + 0) > 0$ gives $x < -4$ or $x > 0$.

Linear: $-2x + 8 \geq 11$ gives $x \leq -\frac{3}{2}$. (the inequality reverses as we divide by a negative)

Both must hold: take the overlap of the two sets.
Answer: x < −4

6 Relative motion of two projectiles (AQA, Edexcel, OCR, MEI, Eduqas)

Particles A and B are projected from a point O at the same instant, with initial velocities $(13\mathbf{i} + 33\mathbf{j})$ m s^{-1} and $(8\mathbf{i} + 21\mathbf{j})$ m s^{-1} respectively (i horizontal, j vertically upwards).
Both are still in flight after 2 s. Find (a) the distance AB at this time, (b) the angle AB makes with the horizontal, to 1 d.p.

Position of A: $13t\mathbf{i} + (33t - 4.9t^2)\mathbf{j}$; position of B: $8t\mathbf{i} + (21t - 4.9t^2)\mathbf{j}$
Subtracting, the $4.9t^2$ terms cancel: AB = $t(5\mathbf{i} + 12\mathbf{j})$
After 2 s: AB = $(10\mathbf{i} + 24\mathbf{j})$
Distance = $\sqrt{(10^2 + 24^2)} = 26$

The direction of AB is that of $(5\mathbf{i} + 12\mathbf{j})$, which does not depend on t: tan angle = $\frac{12}{5}$

Answer: (a) 26 m (b) 67.4°

7 Binomial: finding p from a ratio (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X ~ B(8, p). It is given that $P(X = 2) = \frac{3}{2} \times P(X = 1)$.

(a) Find the value of p.
(b) Find P(X = 0), correct to 3 significant figures.

$\frac{56}{2} p^2(1 - p)^6 = \frac{3}{2} \times 8\, p(1 - p)^7$

Divide by $p(1 - p)^6$: $3.5\, p = \frac{3}{2}(1 - p)$

$(3.5 + \frac{3}{2})p = \frac{3}{2}$

$p = \frac{3}{10}$

$P(X = 0) = (1 - p)^8 = (\frac{7}{10})^8 = 0.0576$

Answer: (a) $p = \frac{3}{10}$ (b) 0.0576

8 Trigonometric integrals with identities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use a suitable trigonometric identity to find the exact value of

$\int_0^{\pi/4} \sin 4x \cos x \, dx$

Use $2 \sin A \cos B \equiv \sin(A + B) + \sin(A - B)$
Integrate term by term
Substitute the limits

Answer: $\frac{4}{15} + \frac{2}{15}\sqrt{2}$

9 Ladder against a smooth wall: how high can a person climb (AQA, Edexcel, OCR, MEI)

A uniform ladder AB of length 10 m and weight 150 N has A on rough ground ($\mu = 0.6$) and B against a smooth vertical wall. It makes angle θ with the ground, $\tan\theta = \frac{4}{3}$.

A man of weight 600 N climbs the ladder. How far along the ladder can he go before it slips?

Wall reaction S horizontal, ground reaction R vertical, friction F = S. Moments about A: $S \times L \sin\theta = W \times \frac{L}{2} \cos\theta$

So $S = \frac{W}{2\tan\theta}$. Unloaded, R = W: need $\frac{W}{2\tan\theta} \le \mu W$, so $\mu \ge \frac{1}{2 \times \frac{4}{3}} = 0.375$

With the man at distance x: R = 750, and moments about A give $S L \sin\theta = (150 \times 5 + 600x)\cos\theta$

Limiting: $S = \mu R = 0.6 \times 750 = 450$, so $450 \times 10 \times \frac{4}{3} = 150 \times 5 + 600x$

$600x = 6000 - 750$, so $x = 8.75$

Answer: 8.75 m

10 Distribution of the larger of two scores (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Two fair 6-sided dice, numbered 1 to 6, are rolled. M is the larger of the two scores.

(a) Show that $P(M = m) = \frac{(2m - 1)}{36}$ for m = 1, 2, ..., 6.

(b) Find P(M = 3) and E(M).

(a) $M \leq m$ means both scores $\leq m$: $P(M \leq m) = (m/6)^2$, so $P(M = m) = \frac{m^2 - (m - 1)^2}{36} = \frac{(2m - 1)}{36}$

(b) $P(M = 3) = \frac{5}{36}$

$E(M) = \Sigma\, m\, P(M = m) = \frac{1}{36} \Sigma\, m(2m - 1)$

$= \frac{161}{36}$

Answer: (b) $P(M = 3) = \frac{5}{36}$, $E(M) = \frac{161}{36}$