Day 289 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Projectile through a given point: two angles (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground at 35 m s^{-1}, at angle θ above the horizontal. It passes through the point P, 60 m horizontally and 4 m vertically from O.

Find the two possible values of $\tan\theta$, and the two possible values of θ to 1 d.p. Take g = 9.8 m s^{-2}.

Horizontal: $x = 35\cos\theta \times t$. Vertical: $y = 35\sin\theta \times t - 4.9t^2$

Eliminating t: $y = x\tan\theta - \frac{9.8x^2}{2u^2}(1 + \tan^2\theta)$

$\frac{9.8 \times 60^2}{2 \times 35^2} = 14.4$, so $4 = 60\tan\theta - 14.4(1 + \tan^2\theta)$

$14.4\tan^2\theta - 60\tan\theta + 18.4 = 0$

$\tan\theta = \frac{1}{3}$ or $\frac{23}{6}$, so $\theta = 18.4°$ or $75.4°$

Answer: $\tan\theta = \frac{1}{3}$ or $\frac{23}{6}$; $\theta = 18.4°$ or $75.4°$

2 Circle through three points (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A circle passes through A(−10, 13), B(6, 5) and C(4, −1). Find its equation.

Centre is on the perpendicular bisector of each chord.
Bisector of AB: $16x - 8y = -104$.
Bisector of BC: $-2x - 6y = -22$.
Solving: centre $(-4, 5)$.
$r^2 = (-10 - (-4))^2 + (13 - 5)^2 = 100$.
Answer: $(x + 4)^2 + (y - 5)^2 = 100$

3 Inverse trigonometric values (Edexcel, OCR, MEI, Eduqas, CIE (inverse trig))

Find the exact value of

$\tan(\tan^{-1}(\frac{2}{9}) + \tan^{-1}(\frac{1}{4}))$.

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$= \frac{\frac{2}{9} + \frac{1}{4}}{1 - \frac{2}{9} \times \frac{1}{4}} = \frac{\frac{17}{36}}{\frac{17}{18}}$

Answer: $\frac{1}{2}$

4 Range and greatest height to speed and angle (AQA, Edexcel, OCR, MEI, Eduqas)

A golf ball is hit from a point O on horizontal ground. It lands 32 m from O, and reaches a maximum height of 20 m above the ground.

Find the angle of projection and the speed of projection, to 3 s.f. Take g = 9.8 m s^{-2}.

Vertical: $0 = u_y^2 - 2 \times 9.8 \times 20$, so $u_y = 19.80$

Time of flight = $\frac{2u_y}{9.8} = 4.041$ s, so $u_x = \frac{32}{4.041} = 7.920$

$\tan\theta = \frac{u_y}{u_x} = 2.500$, so $\theta = 68.2°$

Speed = $\sqrt{(u_x^2 + u_y^2)} = 21.3$

Answer: $\theta = 68.2°$, speed 21.3 m s^{-1}

5 Parametric differentiation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = 5\cos t$, $y = 3\sin t$.
Find a cartesian equation of the curve, with no fractions.

$\cos t = \frac{x}{5}$, $\sin t = \frac{y}{3}$

Use $\cos^2 t + \sin^2 t = 1$: $\frac{x^2}{25} + \frac{y^2}{9} = 1$

Multiply by 225

Answer: $9x^2 + 25y^2 = 225$

6 Hypothesis test for a normal mean (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The daily output of a machine is $N(\mu, 9)$ with known variance 9. The mean was claimed to be 46 units.
A random sample of 64 has mean 47. Test at the 5% level whether the mean is different from 46.

$H_0: \mu = 46$, $H_1: \mu \neq 46$. Under H_0, the sample mean ~ $N(46, \frac{9}{64})$

$z = \frac{47 - 46}{3 \div \sqrt{64}} = 2.667$

One tail: $P(Z > 2.667) = 0.00383$, compared with 2.5%
$0.00383 < 2.5\%$, so reject H_0
There is sufficient evidence at the 5% level that the mean has changed as claimed
Answer: Reject H_0 ($z = 2.667$, p-value 0.00383)

7 Equations in disguise (substitution) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $x - 8\sqrt{x} + 12 = 0$.

Let $u = \sqrt{x}$: $u^2 - 8u + 12 = 0$.
$(u - 2)(u - 6) = 0$, so $\sqrt{x} = 2$ or $\sqrt{x} = 6$.
Square: $x = 4$ or $x = 36$.
Answer: $x = 4$ or $x = 36$

8 Poisson: scaling the rate, then binomial (CIE)

Calls arrive at a switchboard at an average rate of 12 per hour, independently and at random.
(a) Find the probability of at least 3 in a 15-minute period.
(b) Find the probability that, of 8 separate 15-minute periods, exactly 3 have no arrivals.

15-minute period: $\lambda = 12 \times \frac{15}{60} = 3$, so $X \sim Po(3)$

(a) $P(X \geq 3) = 1 - P(X \leq 2) = 0.5768$
(b) $P(X = 0) = e^{-3} = 0.04979$
$Y \sim B(8, 0.04979)$: $P(Y = 3) = 0.00535$
Answer: (a) 0.577 (b) 0.00535

9 Implicit curves: horizontal and vertical tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^2 + xy + 5y^2 = 19$.
Find the coordinates of the points on C where the tangent is parallel to the x-axis.

Differentiate: $2x + by + (bx + 2cy)dy/dx = 0$, so $dy/dx = -\frac{2x + y}{x + 10y}$

Horizontal tangent when $2x + y = 0$, so $y = -\frac{2x}{1}$

Substitute into the equation of C: $x^2 + (xy) + (5y^2) = 19$
Solve for the remaining coordinate
Answer: $(1, -2)$ and $(-1, 2)$

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10 Geometric series: least n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A geometric series has first term 7 and common ratio $\frac{1}{2}$.

Find the least n for which S_n exceeds 99% of the sum to infinity.

$S_n = S_\infty(1 - r^n) > 0.99\, S_\infty$.

$1 - \frac{1}{2}^n > 0.99$, so $\frac{1}{2}^n < 0.01$.

$n > \frac{\log(0.01)}{\log(0.5)} = 6.644$.

So the least n is 7.

Answer: n = 7