Day 288 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Boat crossing a river (AQA, Edexcel, OCR, MEI)

A river is 150 m wide and flows at 17 m s^{-1}. A boat has speed 8 m s^{-1} in still water and crosses the river heading so that it is carried as little distance downstream as possible.
Find the least possible distance downstream between the starting point and the landing point.

The boat speed is less than the current, so it cannot stop drifting. Let the boat head at angle φ upstream from the perpendicular to the bank

Across: 8 cos φ; along (downstream): 17 − 8 sin φ. Drift = $\frac{150}{8\cos\varphi}(17 - 8\sin\varphi)$

Drift is least when the resultant velocity is at the greatest angle to the downstream direction: the boat velocity is perpendicular to the resultant, so $\sin\varphi = \frac{8}{17}$

Then across speed = $\frac{120}{17}$ and downstream speed = $\frac{225}{17}$, so drift = $150 \times \frac{\frac{225}{17}}{\frac{120}{17}}$

Answer: 281.25 m

2 Stationary points of x^n times a root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = 2x\sqrt{3 - x}$, for $0 < x < 3$, has one stationary point. Find its exact coordinates and determine its nature.

$dy/dx = \frac{2(6 - 3x)}{2\sqrt{3 - x}}$
$dy/dx = 0$: $6 - 3x = 0$, so $x = 2$
$y = 2 \times 2 \times 1$
Gradient changes from positive to negative: a maximum
Answer: (2, 4), maximum

3 Polynomial inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $x^3 - 3x - 2 > 0$.

Use the factor theorem to find roots: −1, 2.
$f(x) = (x + 1)^2(x - 2)$.
Sign of f in each region (left to right): −, −, +.
Take the regions where $f > 0$ (include roots if ≤ or ≥).
Answer: $x > 2$

4 Simultaneous logarithmic equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve the simultaneous equations
$\lg x - \lg y = 2$
$x + y = 101$

$\lg(x/y) = 2$, so $x = 10^2 y = 100y$
$100y + y = 101$, so $101y = 101$
Answer: $x = 100$, $y = 1$

5 Least number of trials (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each bulb is faulty with probability 0.05, independently.
Find the least number n of bulbs that must be tested so that the probability of finding at least one faulty is greater than 0.9.

P(at least one faulty) = $1 - 0.95^n > 0.9$
$0.95^n < 0.1$
$n \ln 0.95 < \ln 0.1$
Dividing by ln 0.95 (negative) reverses the inequality: $n > 44.89$
$n = 45$
Answer: $n = 45$

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6 Two particles over a pulley: height reached (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particles of mass 12 kg and 8 kg hang on a light string over a smooth fixed pulley. The 12 kg particle is 1.5 m above the ground; the 8 kg one is on the ground.

Released from rest, find (a) the tension, (b) the greatest height reached by the 8 kg particle. Take g = 9.8 m s^{-2}.

$12\,g - T = 12\,a$ and $T - 8\,g = 8\,a$, so $a = \frac{12-8}{12+8} \times 9.8 = 1.96$

$T = 12(9.8 - 1.96) = 94.08$

The 8 kg particle rises 1.5 m before the 12 kg particle lands, reaching $v^2 = 2 \times 1.96 \times 1.5 = 5.88$

The string goes slack, so it then moves freely under gravity: extra height $= \frac{v^2}{2g} = \frac{5.88}{19.6} = 0.3$

Greatest height = 1.5 + 0.3

Answer: (a) 94.08 N (b) 1.8 m

7 Reciprocal graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the curve C, its asymptotes (dashed) and a point P. C has equation $y = \frac{a}{x+p} + q$. Find a, p and q.

Asymptotes $x = 1$ and $y = 2$, so $p = -1$ and $q = 2$.

$y = \frac{a}{x-1} + 2$.

P(3, −1): $-1 = \frac{a}{2} + 2$, so $a = -6$.

Answer: a = −6, p = −1, q = 2

8 Trigonometric integrals with identities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use a suitable trigonometric identity to find the exact value of

$\int_0^{\pi/3} \sin 5x \cos x \, dx$

Use $2 \sin A \cos B \equiv \sin(A+B) + \sin(A-B)$

Integrate term by term

Substitute the limits

Answer: $\frac{3}{16}$

9 Binomial expansion (integer powers) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the term independent of x in the expansion of $(3x - \frac{3}{x^2})^6$.

General term: $C(6,r)(3x)^{6-r}(-\frac{3}{x^2})^r$.

Power of x: $1(6 - r) - 2r = 0$, so $r = 2$.

Term = $C(6,2) \times 3^4 \times (-3)^2 = 10935$.

Answer: 10935

10 Shaded normal curve and the standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The pdf of $X \sim N(95, \sigma^2)$ is shown; the shaded area is symmetrical.

(a) Find σ. (b) Find P(X > 109).

The area 0.95 leaves 0.050 in total in the tails, 0.025 in each

P(X < 85) = 0.025 so $\frac{85-95}{\sigma} = -1.9600$

$\sigma = \frac{10}{1.9600} = 5.102$

P(X > 109) = P(Z > 2.744) = 0.00304

Answer: (a) 5.10 (b) 0.00304