Day 287 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Equations quadratic in an exponential (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $5^{2x} - 4 \times 5^x + 3 = 0$,
giving each answer exactly and, where not an integer, to 3 s.f.

Let $u = 5^x$: $u^2 - 4u + 3 = 0$

$(u - 3)(u - 1) = 0$, so $5^x = 3$ or $5^x = 1$

Take logarithms: $x = \log_5 1 = 0$; $x = \log_5 3 = 0.683$

Answer: $x = \log_5 1 = 0$; $x = \log_5 3 = 0.683$

2 Sums of multiples (arithmetic series) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the sum of all the integers from 1 to 793
inclusive that are divisible by 3 or by 7, but not by both.

Multiples of 3: 264 terms (3 to 792): sum = 104940.
Multiples of 7: 113 terms: sum = 45087.
Multiples of both, i.e. of 21: 37 terms: sum = 14763.
Required sum = 104940 + 45087 − 2 × 14763.
Answer: 120501

3 Car on a hill: power and maximum speed (CIE)

A car of mass 1500 kg, with engine power 29.7 kW, climbs a straight road with $\sin\alpha = \frac{1}{25}$ against a constant resistance of 600 N.

Find (a) the greatest steady speed up the hill, (b) the acceleration at $\frac{25}{3}$ m s^{-1}. Take g = 9.8 m s^{-2}.

Steady speed: driving force = $600 + 1500 \times 9.8 \times \frac{1}{25} = 1188$ N

$P = F v$, so $v = \frac{29700}{1188} = 25$

At $\frac{25}{3}$ m s^{-1}: $F = \frac{29700}{\frac{25}{3}} = 3564$ N

$F - 1188 = 1500 a$, so $a = \frac{2376}{1500}$

Answer: (a) 25 m s^{-1} (b) 1.58 m s^{-2}

4 Normal: find mean and standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The scores in an aptitude test are normally distributed. 5% are less than 17 and 25% are greater than 31.
Find the mean and standard deviation, each correct to 3 significant figures.

Let $X \sim N(\mu, \sigma^2)$. $P(X < 17) = 0.05$ gives $\frac{17 - \mu}{\sigma} = -1.6449$

$P(X > 31) = 0.25$ gives $\frac{31 - \mu}{\sigma} = 0.6745$

Subtract: $\frac{14}{\sigma} = 0.6745 - (-1.6449) = 2.3193$

$\sigma = 6.036$

$\mu = 17 - (-1.6449)(6.036) = 26.93$

Answer: $\mu = 26.9$, $\sigma = 6.04$

5 Force applied then removed (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of mass 25 kg is at rest on rough horizontal ground ($\mu = 0.5$). A horizontal force of 140 N acts on it for 7 s and is then removed.

Find (a) the speed when the force is removed, (b) the total distance travelled before the box stops. Take $g = 9.8$ m s^{-2}.

Friction = $\mu mg = 0.5 \times 25 \times 9.8 = 122.5$ N

While pushed: $140 - 122.5 = 25a$, so $a = 0.7$

$v = 0.7 \times 7 = 4.9$; $s_1 = \frac{1}{2} \times 0.7 \times 7^2 = 17.15$

After release: deceleration = $\mu g = 4.9$, so $s_2 = \frac{4.9^2}{2 \times 4.9} = 2.45$

Total distance = 17.15 + 2.45

Answer: (a) 4.9 m s^{-1} (b) 19.6 m

6 Implicit differentiation: tangent or normal (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^2y + xy^2 = -6$.

Find the equation of the tangent to C at the point $(-1, -2)$, in the form $ax + by + c = 0$ with integers a, b, c.

Differentiate implicitly with respect to x

At $(-1, -2)$: $\frac{dy}{dx} = -\frac{8}{5}$

$y - -2 = (-\frac{8}{5})(x - -1)$, then clear fractions

Answer: 8x + 5y + 18 = 0

7 Self-inverse function (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{kx + 9}{x - 4}$ for $x \neq 4$. Given that f is self-inverse, find k, and hence solve $f(x) = x$.

Let $y = f(x)$: $y(x - 4) = kx + 9$, so $x(y - k) = 4y + 9$.

$f^{-1}(x) = \frac{4x + 9}{x - k}$.

$f = f^{-1}$ requires $k = 4$.

$f(x) = x$: $4x + 9 = x(x - 4)$, so $x^2 - 8x - 9 = 0$.

$x = 4 \pm \sqrt{25} = 4 \pm 5$.

Answer: k = 4; x = 9 or x = −1

8 Simultaneous equations (linear and quadratic) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously:

$x - 2y = -2$

$x^2 - xy - y^2 = 4$

From the first equation $x = -2 + 2y$.

Substitute into the second equation and collect terms:

this gives $y^2 - 6y = 0$.

$y = 0$ or $y = 6$, giving $x = -2$ or $x = 10$.

Answer: (x, y) = (−2, 0) and (10, 6)

9 Closest approach of moving points (Edexcel, OCR, MEI, Eduqas, CIE (vectors in kinematics))

Two boats P and Q move so that, t hours after noon, their position vectors in km are

$\mathbf{r}_P = \begin{pmatrix}2\\0\end{pmatrix} + t\begin{pmatrix}-1\\1\end{pmatrix}$ and $\mathbf{r}_Q = \begin{pmatrix}-8\\2\end{pmatrix} + t\begin{pmatrix}1\\1\end{pmatrix}$.

Find the time at which they are closest and the exact minimum distance between them.

$\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix}-10\\2\end{pmatrix} + t\begin{pmatrix}2\\0\end{pmatrix}$

Distance$^2 = |\mathbf{r}_Q - \mathbf{r}_P|^2$, minimised when $(\mathbf{r}_Q - \mathbf{r}_P) \cdot \begin{pmatrix}2\\0\end{pmatrix} = 0$

$-20 + t(4) = 0$, so $t = 5$

Then $\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix}0\\2\end{pmatrix}$, with length $\sqrt{4}$

Answer: t = 5, minimum distance = 2 km

10 Sum of two independent copies of a discrete variable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X takes values 2, 3, 4, 5 with $P(X = x) = k(8 - x)$. X_1 and X_2 are independent copies of X.

(a) Find k.

(b) Find $P(X_1 + X_2 = 8)$.

(c) Find $P(X_1 = X_2)$ and $E(X_1 + X_2)$.

(a) $\Sigma P = k(6 + 5 + 4 + 3) = 18k = 1$, so $k = \frac{1}{18}$

(b) Ways to total 8: (3,5), (4,4), (5,3)

$P = \frac{5}{18}\times\frac{1}{6} + \frac{2}{9}\times\frac{2}{9} + \frac{1}{6}\times\frac{5}{18} = \frac{23}{162}$

(c) $P(X_1 = X_2) = \Sigma [P(X = x)]^2 = \frac{43}{162}$

$E(X) = \frac{29}{9}$, so $E(X_1 + X_2) = 2 \times \frac{29}{9} = \frac{58}{9}$

Answer: (a) $k = \frac{1}{18}$ (b) $\frac{23}{162}$ (c) $\frac{43}{162}$, $\frac{58}{9}$