Day 286 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Outliers and the 1.5 × IQR rule (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A data set has lower quartile 30 and upper quartile 43. An outlier is a value more than 1.5 × IQR beyond the nearest quartile.
(a) Find the range of values that are not outliers.
(b) 12 observations have mean 33. The largest is 73, an outlier, and it is removed. Find the new mean.

IQR = 43 − 30 = 13, so 1.5 × IQR = 19.5
Lower limit = 30 − 19.5 = 10.5, upper limit = 43 + 19.5 = 62.5
$\Sigma x = 12 \times 33 = 396$, so without 73: $\Sigma x = 323$
New mean = $\frac{323}{11}$

Answer: (a) 10.5 to 62.5 (b) 29.36

2 Dropped and projected particles meet (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Ball A is projected vertically upwards from ground level with speed 30 m s^{-1}. At the same instant, ball B is released from rest 30 m directly above A.
The balls collide. Find (a) the height above the ground at which they collide, (b) the velocity of A just before the collision.
Take g = 9.8 m s^{-2}.

Relative to B, A moves upwards at the constant speed 30 m s^{-1} (both have acceleration g)
So the balls meet when 30 t = 30, giving t = 1
Height of A: $s = 30 \times 1 - \frac{1}{2} \times 9.8 \times 1^2 = 25.1$
$v = u - gt = 30 - 9.8 \times 1 = 20.2$

Answer: (a) 25.1 m (b) 20.2 m s^{-1} upwards

3 Partial fractions (repeated factor) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{-x^2 + 7x - 16}{(x-4)(x-3)^2}$ in the form
$\frac{A}{(x-4)} + \frac{B}{(x-3)} + \frac{C}{(x-3)^2}$.

$-x^2 + 7x - 16 \equiv A(x-3)^2 + B(x-4)(x-3) + C(x-4)$.
Put x = 4: $-4 = A(1)^2$, so A = −4.
Put x = 3: −4 = C(−1), so C = 4.
Compare x^2: −1 = 1A + 1B, so B = 3.

Answer: A = −4, B = 3, C = 4

4 sec, cosec and tan/cot quadratics (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $12\text{cosec}^2\theta - 11\cot\theta - 10 = 0$
for $0 \leq \theta < 360°$, giving answers to 1 d.p.

Use $\text{cosec}^2\theta \equiv 1 + \cot^2\theta$: $12\cot^2\theta - 11\cot\theta + 2 = 0$
$(4\cot\theta - 1)(3\cot\theta - 2) = 0$
$\cot\theta = \frac{1}{4}$ or $\cot\theta = \frac{2}{3}$
$\tan\theta = 4$ or $\frac{3}{2}$, so $\theta = 76.0°$ and $56.3°$ (+180°)

Answer: θ = 56.3°, 76.0°, 236.3°, 256.0°

5 PMCC from summary data and a hypothesis test (AQA, Edexcel, OCR, MEI, Eduqas)

For 10 pairs of values, $S_{xx} = 49$, $S_{yy} = 64$ and $S_{xy} = 23$.

(a) Calculate the product moment correlation coefficient r.

(b) The critical value for a one-tailed test at the 1% level with n = 10 is 0.7155. Carry out the test of H_0: $\rho = 0$ against H_1: $\rho > 0$.

$r = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}} = \frac{23}{\sqrt{(49 \times 64)}} = \frac{23}{56} = 0.4107$

H_0: $\rho = 0$, H_1: $\rho > 0$

$|r| = 0.4107 <$ the critical value 0.7155

Do not reject H_0: there is insufficient evidence of positive correlation between the variables

Answer: (a) r = 0.411 (b) do not reject H_0

6 Reciprocal graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the curve C, its asymptotes (dashed) and a point P. C has equation $y = \frac{a}{x + p} + q$. Find a, p and q.

Asymptotes $x = -3$ and $y = -1$, so $p = 3$ and $q = -1$.

$y = \frac{a}{x + 3} - 1$.

P(−4, 5): $5 = \frac{a}{-1} - 1$, so $a = -6$.

Answer: a = −6, p = 3, q = −1

7 R-form: greatest and least values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(\theta) = \frac{2}{31 + 20 \sin\theta + 21 \cos\theta}$

(a) Write $20 \sin\theta + 21 \cos\theta$ as $R \sin(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.

(b) Find the greatest and least values of $f(\theta)$, and the smallest positive θ at which the greatest value occurs, to 1 d.p.

$R = \sqrt{20^2 + 21^2} = 29$, $\tan\alpha = \frac{21}{20}$, $\alpha = 46.4°$

Denominator = $31 + 29 \sin(\theta + 46.4°)$, which lies between 2 and 60

Greatest $f(\theta) = \frac{2}{2} = 1$, when $\sin(\theta + \alpha) = -1$

Least $f(\theta) = \frac{2}{60} = \frac{1}{30}$, when $\sin(\theta + \alpha) = 1$

$\sin(\theta + \alpha) = -1$: $\theta + 46.4° = 270°$, so $\theta = 223.6°$

Answer: (a) $29 \sin(\theta + 46.4°)$ (b) greatest 1, least $\frac{1}{30}$, $\theta = 223.6°$

8 Relative motion of two projectiles (AQA, Edexcel, OCR, MEI, Eduqas)

Particles A and B are projected from a point O at the same instant, with initial velocities $(13\mathbf{i} + 31\mathbf{j})$ m s^{-1} and $(17\mathbf{i} + 34\mathbf{j})$ m s^{-1} respectively (i horizontal, j vertically upwards).

Both are still in flight after 4 s. Find (a) the distance AB at this time, (b) the angle AB makes with the horizontal, to 1 d.p.

Position of A: $13t\mathbf{i} + (31t - 4.9t^2)\mathbf{j}$; position of B: $17t\mathbf{i} + (34t - 4.9t^2)\mathbf{j}$

Subtracting, the $4.9t^2$ terms cancel: $AB = t(-4\mathbf{i} - 3\mathbf{j})$

After 4 s: $AB = (-16\mathbf{i} - 12\mathbf{j})$

Distance $= \sqrt{(16^2 + 12^2)} = 20$

The direction of AB is that of $(-4\mathbf{i} - 3\mathbf{j})$, which does not depend on t: tan angle $= \frac{3}{4}$

Answer: (a) 20 m (b) 36.9°

9 Implicit curves: horizontal and vertical tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^2 + xy + y^2 = 12$.
Find the coordinates of the points on C where the tangent is parallel to the y-axis.

Differentiate: $2x + by + (bx + 2cy)dy/dx = 0$, so $dy/dx = -\frac{2x + y}{x + 2y}$

Vertical tangent when $x + 2y = 0$, so $x = -\frac{2y}{1}$

Substitute into the equation of C: $x^2 + (xy) + (y^2) = 12$
Solve for the remaining coordinate
Answer: $(-4, 2)$ and $(4, -2)$

10 Binomial expansion with partial fractions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{-12x + 1}{(1 - 2x)(1 + 3x)}$.
(a) Express f(x) in partial fractions.
(b) Hence expand f(x) in ascending powers of x up to x^3, stating the range of validity.

(a) $-12x + 1 \equiv A(1 + 3x) + B(1 - 2x)$: $A = -2$, $B = 3$.
$f(x) = \frac{-2}{1 - 2x} + \frac{3}{1 + 3x}$.

(b) $\frac{1}{1 + cx} = 1 - cx + c^2x^2 - c^3x^3 + ...$ (use $c = -2$, then $c = 3$).

Coefficients of x^j: $-2(2)^j + 3(-3)^j$.

Valid for $|x| < \frac{1}{3}$ (both expansions).

Answer: (a) $A = -2$, $B = 3$ (b) $1 - 13x + 19x^2 - 97x^3 + ...$; $|x| < \frac{1}{3}$