Day 285 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Newton-Raphson iteration (AQA, Edexcel, OCR, MEI, Eduqas (Newton-Raphson))

$f(x) = x^3 + x^2 - 6x - 11$.
(a) Show that $f(x) = 0$ has a root α with $2 < \alpha < 3$.
(b) Use Newton-Raphson with $x_0 = 2$ to find x_1 and x_2 to 4 decimal places.

$f(2) = -11$ and $f(3) = 7$: a change of sign, f continuous

$f'(x) = 3x^2 + 2x - 6$, so $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$x_1 = 2 - \dfrac{-11.00}{10.00} = 3.1000$

$x_2 = 3.1000 - \dfrac{9.801}{29.03} = 2.7624$

Answer: $x_1 = 3.1000$, $x_2 = 2.7624$

2 Reciprocal graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the curve C, its asymptotes (dashed) and a point P. C has equation $y = \dfrac{a}{x + p} + q$. Find a, p and q.

Asymptotes $x = -3$ and $y = 2$, so $p = 3$ and $q = 2$.

$y = \dfrac{a}{x + 3} + 2$.

P$(-2, -2)$: $-2 = \dfrac{a}{1} + 2$, so $a = -4$.

Answer: $a = -4$, $p = 3$, $q = 2$

3 Differentiating a^x (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact gradient of the curve $y = 7^{4x - 2}$ at the point where $x = \dfrac{3}{2}$.

$y = e^{(4x - 2)\ln 7}$
$dy/dx = 4 \ln 7 \times 7^{4x - 2}$
At $x = \dfrac{3}{2}$: exponent = 4, so $dy/dx = 4 \ln 7 \times 7^4$

Answer: $9604 \ln 7$

4 Three-stage journey (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A train accelerates uniformly from rest at 1 m s^{-2} to a speed of V m s^{-1}, travels at this speed, then decelerates uniformly at 1.5 m s^{-2} to rest.
The journey takes 56 s and covers 552 m. Find V and the time spent at constant speed.

Time accelerating = V, time decelerating = $\dfrac{V}{1.5}$, so time at constant speed = $56 - V - \dfrac{V}{1.5}$

Area under the speed-time graph: $552 = 56V - \dfrac{V^2}{2} - \dfrac{V^2}{3}$

Multiply by 2ab: $2.5V^2 - 168V + 1656 = 0$
$V = 12$ or $V = 55.2$
$V = 55.2$ gives time at constant speed -36 s, which is negative, so $V = 12$
Time at constant speed = $56 - 12 - 8 = 36$ s
Answer: V = 12, 36 s

5 Transformations of y = x^2 (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Describe a sequence of two transformations that maps the curve $y = x^2$ onto the curve $y = -2x^2 - 20x - 53$.

Complete the square: $y = -2(x + 5)^2 - 3$.

Step 1: $y = x^2 \rightarrow y = -2x^2$: a stretch, scale factor 2, parallel to the y-axis and a reflection in the x-axis.

Step 2: replace x by (x + 5) and add −3: translation by $\begin{pmatrix} -5 \\ -3 \end{pmatrix}$.

Answer: a stretch, scale factor 2, parallel to the y-axis and a reflection in the x-axis, then translation $\begin{pmatrix} -5 \\ -3 \end{pmatrix}$

6 Conditional probability from given values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

For events A and B, P(A) = 0.4, P(B) = 0.55 and P(A | B) = 0.15.

(a) Find P(A ∩ B).

(b) Find P(A | B′).

$P(A \cap B) = P(A \mid B) \times P(B) = 0.15 \times 0.55 = 0.0825$

$P(A \cup B) = 0.4 + 0.55 - 0.0825 = 0.8675$

$P(B') = 0.45$

$P(A \cap B') = P(A) - P(A \cap B) = 0.3175$

$P(A \mid B') = \frac{0.3175}{0.45}$

Answer: (a) 0.0825 (b) $\frac{127}{180}$

7 Recurrence: limit and convergence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = -\frac{1}{3}u_n + 20$, $u_1 = 3$.

(a) Find the limit L of the sequence.

(b) Find the least n such that $|u_n - L| < 0.01$.

(a) At the limit $L = -\frac{1}{3}L + 20$, so $L(1 - (-\frac{1}{3})) = 20$.

L = 15.

(b) $u_n - L = (u_1 - L)((-\frac{1}{3}))^{n-1} = (-12)((-\frac{1}{3}))^{n-1}$.

Need $12 \times (\frac{1}{3})^{n-1} < 0.01$, so $n - 1 > 6.454$.

The least n is 8.

Answer: (a) L = 15 (b) n = 8

8 R-form: greatest and least values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(\theta) = \frac{4}{19 + 5\sin\theta + 12\cos\theta}$

(a) Write $5\sin\theta + 12\cos\theta$ as $R\sin(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.

(b) Find the greatest and least values of f(θ), and the smallest positive θ at which the greatest value occurs, to 1 d.p.

$R = \sqrt{5^2 + 12^2} = 13$, $\tan\alpha = \frac{12}{5}$, $\alpha = 67.4°$

Denominator = $19 + 13\sin(\theta + 67.4°)$, which lies between 6 and 32

Greatest $f(\theta) = \frac{4}{6} = \frac{2}{3}$, when $\sin(\theta + \alpha) = -1$

Least $f(\theta) = \frac{4}{32} = \frac{1}{8}$, when $\sin(\theta + \alpha) = 1$

$\sin(\theta + \alpha) = -1$: $\theta + 67.4° = 270°$, so $\theta = 202.6°$

Answer: (a) $13\sin(\theta + 67.4°)$ (b) greatest $\frac{2}{3}$, least $\frac{1}{8}$, $\theta = 202.6°$

9 Alternate turns and sum to infinity (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A and B take turns, A first, to score; the first to score wins. P(A scores) = $\frac{1}{6}$, P(B scores) = $\frac{1}{6}$ each attempt.

Find P(A wins (a) on attempt 1 or 2, (b) overall).

P(A wins at attempt 1) = $\frac{1}{6}$; to reach attempt 2 both miss first: $(\frac{5}{6})(\frac{5}{6}) = \frac{25}{36}$

(a) $\frac{1}{6} + \frac{25}{36} \times \frac{1}{6} = \frac{61}{216}$

(b) P(A wins) = $\frac{1}{6} + \frac{25}{36} \times \frac{1}{6} + \frac{25}{36}^2 \times \frac{1}{6} + ...$

Geometric: first term $\frac{1}{6}$, common ratio $\frac{25}{36}$

$S = \frac{a}{1-r} = \frac{\frac{1}{6}}{\frac{11}{36}} = \frac{6}{11}$

Answer: (a) $\frac{61}{216}$ (b) $\frac{6}{11}$

10 Hinged rod supported by a string (AQA, Edexcel, OCR, MEI)

A uniform rod AB (2 m, 60 N) is hinged at A to a vertical wall and held horizontal by a string BC, C on the wall above A, with $\tan \alpha = \frac{5}{12}$, where α is the angle between string and rod. A 50 N particle hangs at P, AP = 1.5 m.

Find (a) the tension, (b) the magnitude and direction of the hinge force (3 s.f. and 1 d.p.).

$\sin \alpha = \frac{5}{13}$, $\cos \alpha = \frac{12}{13}$. Moments about A: $T \sin \alpha \times 2 = 60 \times 1 + 50 \times 1.5$

$T \times \frac{5}{13} \times 2 = 135$, so $T = 175.5$

Horizontal: $H = T \cos \alpha = 162$. Vertical: $V + T \sin \alpha = 60 + 50$, so $V = 42.5$

Force = $\sqrt{(162^2 + 42.5^2)} = 167$, angle = $\tan^{-1}(\frac{42.5}{162}) = 14.7°$ above the horizontal

Answer: (a) 175.5 N (b) 167 N at 14.7° above the horizontal (away from the wall)