Day 284 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Factor theorem and cubic (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = 3x^3 + ax^2 + bx - 36$, where a and b are constants.
$(x + 3)$ is a factor of $f(x)$, and $f(x)$ leaves remainder 0 when divided by $(x + 2)$.
Find a and b, and hence solve $f(x) = 0$.

$f(-3) = 0$: $-81 + 9a - 3b - 36 = 0$, so $9a - 3b - 117 = 0$.
$f(-2) = 0$: $4a - 2b - 60 = 0$.
Solving: $a = 9$, $b = -12$.
$f(x) = 3x^3 + 9x^2 - 12x - 36 = 3(x + 3)(x + 2)(x - 2)$
Answer: a = 9, b = −12; x = −3, −2, 2

2 Hypothesis test for a normal mean (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The fill volume of a bottle is $N(\mu, 9)$ with known variance 9. The mean was claimed to be 67 ml.
A random sample of 49 has mean 68. Test at the 5% level whether the mean is less than 67.

H_0: $\mu = 67$, H_1: $\mu < 67$. Under H_0, the sample mean ~ $N(67, \frac{9}{49})$

$z = \frac{68 - 67}{3 \div \sqrt{49}} = 2.333$
p-value = $P(Z < 2.333) = 0.990$
$0.990 > 5\%$, so do not reject H_0
There is insufficient evidence at the 5% level that the mean has changed as claimed
Answer: Do not reject H_0 (z = 2.333, p-value 0.990)

3 Least force to drag a box (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of weight 187 N is pulled at constant speed along rough horizontal ground ($\mu = \frac{8}{15}$) by a force P N at angle θ above the horizontal.

(a) Find P when $\tan\theta = \frac{5}{12}$. (b) Find the least possible value of P as θ varies, and the corresponding θ to 1 d.p.

Vertical: $R + P\sin\theta = 187$. Horizontal: $P\cos\theta = \mu R = \frac{8}{15}(187 - P\sin\theta)$

$P(\cos\theta + \frac{8}{15}\sin\theta) = \frac{8}{15} \times 187$, so $P = \frac{1496}{15} \div (\cos\theta + \frac{8}{15}\sin\theta)$

(a) $\cos\theta = \frac{12}{13}$, $\sin\theta = \frac{5}{13}$: $P = \frac{1496}{15} \div \frac{44}{39} = 88.4$

(b) P is least when $\cos\theta + \frac{8}{15}\sin\theta$ is greatest. Its greatest value is $\sqrt{(1 + \frac{8}{15}^2)} = \frac{17}{15}$,

which occurs when $\tan\theta = \frac{8}{15}$. Then $P = \frac{1496}{15} \div \frac{17}{15} = 88$ and $\theta = 28.1°$

Answer: (a) 88.4 N (b) 88 N at θ = 28.1°

4 Outliers and the 1.5 × IQR rule (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A data set has lower quartile 35 and upper quartile 49. An outlier is a value more than 2 × IQR beyond the nearest quartile.
(a) Find the range of values that are not outliers.
(b) 12 observations have mean 49. The largest is 81, an outlier, and it is removed. Find the new mean.

IQR = 49 − 35 = 14, so 2 × IQR = 28
Lower limit = 35 − 28 = 7, upper limit = 49 + 28 = 77
$\Sigma x = 12 \times 49 = 588$, so without 81: $\Sigma x = 507$
New mean = $\frac{507}{11}$
Answer: (a) 7 to 77 (b) 46.09

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5 Fixed point iteration and rearrangement (AQA, Edexcel, OCR, MEI, Eduqas, CIE (iterative methods))

The equation $x^3 = 5x + 20$ has a root α.
(a) Show that the equation can be written as $x = \sqrt[3]{(5x + 20)}$.
(b) Use $x_{n+1} = \sqrt[3]{(5x_n + 20)}$ with $x_0 = 2$ to find x_1, x_2, x_3
to 4 d.p., and hence state α to 3 d.p.

Take the cube root of both sides to obtain $x = \sqrt[3]{(5x + 20)}$
$x_1 = 3.1072$, $x_2 = 3.2877$, $x_3 = 3.3153$
The iteration converges (successive values agree to 3 d.p.): $\alpha = 3.320$
Answer: $x_1 = 3.1072$, $x_2 = 3.2877$, $x_3 = 3.3153$; $\alpha = 3.320$

6 Partial fractions (improper) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{x^3 + 7x^2 + 16x + 16}{(x + 1)(x + 2)}$ in the form $ax + b + \frac{C}{(x + 1)} + \frac{D}{(x + 2)}$.

Divide: the numerator has degree 3, the denominator degree 2.
$(x^3 + 7x^2 + 16x + 16) \div (x^2 + 3x + 2)$ gives quotient $x + 4$.
Remainder: $2x + 8$.
Put $x = -1$: remainder $= C(1)$ so $C = 6$. Put $x = -2$: $D = -4$.
Answer: $x + 4 + \frac{6}{(x + 1)} - \frac{4}{(x + 2)}$

7 Integration by parts (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use integration by parts to find the exact value of

$\int_0^1 x^2 e^{2x}\, dx$

Parts twice: $u = x^2$, $dv/dx = e^{2x}$
Integral $= [\frac{x^2}{2}e^{2x}] - \int \frac{2x}{2}e^{2x}\, dx$
Repeat with $u = x$ on the remaining integral
Answer: $-\frac{1}{4} + \frac{1}{4}e^2$

8 Polynomial inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $x^4 - 8x^3 + 15x^2 + 8x - 16 > 0$.

Use the factor theorem to find roots: -1, 1, 4.
$f(x) = (x + 1)(x - 1)(x - 4)^2$.
Sign of f in each region (left to right): $+$, $-$, $+$, $+$.
Take the regions where $f > 0$ (include roots if $\leq$ or $\geq$).
Answer: $x < -1$ or $1 < x < 4$ or $x > 4$

9 Optimisation with justification (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Squares of side x cm are cut from the corners of a square sheet of
side 12 cm and the sides folded up to make an open box.
Find the value of x that maximises the volume, and that maximum volume.

$V = x(12 - 2x)^2 = 4x^3 - 48x^2 + 144x$
$dV/dx = 12x^2 - 96x + 144 = (12 - 2x)(12 - 6x) = 0$
$x < 6$, so $x = 2$
$d^2V/dx^2 = 24x - 96$, negative at $x = 2$: maximum
$V = 2 \times 8^2$
Answer: $x = 2$, $V = 128$ cm^3

10 Interception at constant speed (AQA, Edexcel, OCR, MEI, Eduqas)

At time t = 0 hours, ship A is at the point with position vector $(18\mathbf{i} + 26\mathbf{j})$ km relative to a port O, and moves with constant velocity $(6\mathbf{i} - 5\mathbf{j})$ km h^{-1}. Ship B leaves O at t = 0 with constant speed 17 km h^{-1} and intercepts A.
Find (a) the time at which B intercepts A, (b) the velocity of B as a vector.

Position of A at time t: $(18\mathbf{i} + 26\mathbf{j}) + t(6\mathbf{i} - 5\mathbf{j})$. B at time t: t v_B (unknown velocity)
At interception B has travelled 17 t, so $|(18\mathbf{i} + 26\mathbf{j}) + t(6\mathbf{i} - 5\mathbf{j})| = 17\,t$
Squaring and rearranging: $-228t^2 - 44t + 1000 = 0$, which gives t = 2 or a negative value
t = 2 (the other root is negative since B is faster than A)

A is then at $(18\mathbf{i} + 26\mathbf{j}) + 2(6\mathbf{i} - 5\mathbf{j}) = (30\mathbf{i} + 16\mathbf{j})$, so $v_B = \frac{1}{2}(30\mathbf{i} + 16\mathbf{j}) = (15\mathbf{i} + 8\mathbf{j})$

Answer: (a) 2 h (b) $(15\mathbf{i} + 8\mathbf{j})$ km h^{-1}