Day 283 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Modelling y = ab^x and y = ax^n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = ax^n$ passes through (1, 12) and (2, 96).
Find the values of a and n.

$12 = a \times 1^n$ and $96 = a \times 2^n$

Divide: $2^n = 8$ so $n = 3$

$a = 12 \div 1^3 = 12$

Answer: a = 12, n = 3

2 Periodic recurrence relations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sequence is defined by $u_1 = 5$ and

$u_{n+1} = \frac{u_n + 4}{3u_n - 1}$.

Show that the sequence is periodic and find

$\sum_{r=1}^{101} u_r$.

$u_1 = 5$, $u_2 = \frac{9}{14}$, $u_3 = 5$.

$u_3 = u_1$, so the sequence repeats with period 2.

One period sums to $\frac{79}{14}$.

$101 = 2 \times 50 + 1$: 50 full periods plus the first 1 term.

Answer: $\frac{2010}{7}$

3 Expected winnings and a fair game (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A bag has 3 red, 6 blue and 3 green counters. A player takes two without replacement. Two reds win £3, two blues win £6, anything else wins nothing.
(a) Find the expected winnings.
(b) The player pays £1 to play. Find the expected profit.

$P(RR) = \frac{3}{12} \times \frac{2}{11} = \frac{1}{22}$

$P(BB) = \frac{6}{12} \times \frac{5}{11} = \frac{5}{22}$

(a) $E = 3 \times \frac{1}{22} + 6 \times \frac{5}{22} = \frac{3}{2}$

(b) Expected profit $= \frac{3}{2} - 1 = \frac{1}{2}$

Answer: (a) £1.50 (b) £0.500

4 Overtaking with a delayed start (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Car A passes a point O travelling at a constant speed of 13.5 m s^{-1}. 6 s later, car B leaves O from rest with constant acceleration 2 m s^{-2}, travelling in the same direction.
B overtakes A at the point X. Find (a) the time B takes to reach X, (b) the distance OX, (c) the speed of B at X.

Let B travel for t s. A: distance = $13.5(t + 6)$; B: distance = $\frac{1}{2} \times 2 \times t^2$

Equate: $2t^2 - 27t - 162 = 0$, so $(t - 18)(2t + 9) = 0$

$t = 18$ (the other root is negative)

OX = $2 \times 18^2 / 2 = 324$ m, and speed = $2 \times 18 = 36$

Answer: (a) 18 s (b) 324 m (c) 36 m s^{-1}

Day 283 **A Level Maths: S Tier 10-a-day: Model Solutions**

5 Pair of inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the set of values of x for which both
$x^2 - 2x - 35 \leq 0$ and $3x - 3 > 10$.

Quadratic: $(x + 5)(x - 7) \leq 0$ gives $-5 \leq x \leq 7$.

Linear: $3x - 3 > 10$ gives $x > \frac{13}{3}$.

Both must hold: take the overlap of the two sets.

Answer: $\mathbf{\frac{13}{3} < x \leq 7}$

6 PMCC from summary data and a hypothesis test (AQA, Edexcel, OCR, MEI, Eduqas)

For 10 pairs of values, $S_{xx} = 49$, $S_{yy} = 256$ and $S_{xy} = 58$.
(a) Calculate the product moment correlation coefficient r.
(b) The critical value for a two-tailed test at the 5% level with n = 10 is 0.6319. Carry out the test of H_0: $\rho = 0$ against H_1: $\rho \neq 0$.

$r = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}} = \frac{58}{\sqrt{(49 \times 256)}} = \frac{58}{112} = 0.5179$
H_0: $\rho = 0$, H_1: $\rho \neq 0$
$|r| = 0.5179$ < the critical value 0.6319
Do not reject H_0: there is insufficient evidence of correlation between the variables
Answer: (a) r = 0.518 (b) do not reject H_0

7 Implicit curves: horizontal and vertical tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^2 - xy + 5y^2 = 19$.
Find the coordinates of the points on C where the tangent is parallel to the x-axis.

Differentiate: $2x + by + (bx + 2cy)dy/dx = 0$, so $dy/dx = -\frac{2x + -y}{-x + 10y}$

Horizontal tangent when $2x + -y = 0$, so $y = \frac{2x}{1}$

Substitute into the equation of C: $x^2 + (-xy) + (5y^2) = 19$
Solve for the remaining coordinate
Answer: (−1, −2) and (1, 2)

8 Line tangent to a hyperbola (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $y = \frac{-x - 1}{x + 3}$.

(a) Write down the equations of the asymptotes of C.
(b) The line $y = -2x + k$ is a tangent to C.
Find the possible values of k.

(a) Vertical: $x = -3$. As $x \to \infty$, $y \to -1$, so $y = -1$.
(b) $-x - 1 = (-2x + k)(x + 3)$.
$-2x^2 + (k - 5)x + (3k + 1) = 0$.
Tangent: $b^2 - 4ac = 0$, which simplifies to $k^2 + 14k + 33 = 0$.
Answer: (a) x = −3, y = −1 (b) k = −11 or k = −3

9 Exponential times trigonometric (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$y = e^{-2x} \sin 2x$. Show that $dy/dx = e^{-2x}(-2 \sin 2x + 2 \cos 2x)$.
Find the x coordinate of the stationary point with the smallest positive x (3 s.f.), and its nature.

Product rule: $dy/dx = (-2)e^{-2x} \sin 2x + e^{-2x} \times 2 \cos 2x$
$dy/dx = 0$ requires $-2 \sin 2x + 2 \cos 2x = 0$
$\tan 2x = 1$, smallest positive $2x = 0.7854$
$x = 0.393$
Second derivative test (or gradient change): a maximum
Answer: x = 0.393, maximum

10 Non-uniform rod: centre of mass and tilting (AQA, Edexcel, OCR, MEI)

A non-uniform rod AB of length 6 m and mass 4 kg rests on supports at C and D, where AC = 2 m and AD = 4 m, with $R_C = 3R_D$.

Find (a) the distance of the centre of mass from A. A particle of mass m kg at B puts the rod on the point of tilting about D. (b) Find m.

$R_C = 3R_D$. Moments about A: $R_C \times 2 + R_D \times 4 = 4g \times AG$

$R_D(3 \times 2 + 4) = 4g \times AG$ with $R_D(1 + 3) = 4g$, so $AG = \frac{3 \times 2 + 4}{4} = 2.5$

Tilting about D: $R_C = 0$. Moments about D: $mg(6 - 4) = 4g(4 - 2.5)$

$m = \frac{4 \times 1.5}{2} = 3$

Answer: (a) 2.5 m (b) 3